

Roots in the mapping class groups

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Abstract

The purpose of this paper is the study of the roots in the mapping class groups. Let Σ be a compact oriented surface, possibly with boundary, let \mathcal{P} be a finite set of punctures in the interior of Σ , and let $\mathcal{M}(\Sigma, \mathcal{P})$ denote the mapping class group of (Σ, \mathcal{P}) . We prove that, if Σ is of genus 0, then each $f \in \mathcal{M}(\Sigma)$ has at most one m -root for all $m \geq 1$. We prove that, if Σ is of genus 1 and has non-empty boundary, then each $f \in \mathcal{M}(\Sigma)$ has at most one m -root up to conjugation for all $m \geq 1$. We prove that, however, if Σ is of genus ≥ 2 , then there exist $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $f^2 = g^2$, f is not conjugate to g , and none of the conjugates of f commutes with g . Afterwards, we focus our study on the roots of the pseudo-Anosov elements. We prove that, if $\partial\Sigma \neq \emptyset$, then each pseudo-Anosov element $f \in \mathcal{M}(\Sigma, \mathcal{P})$ has at most one m -root for all $m \geq 1$. We prove that, however, if $\partial\Sigma = \emptyset$ and the genus of Σ is ≥ 2 , then there exist two pseudo-Anosov elements $f, g \in \mathcal{M}(\Sigma)$ (explicitely constructed) such that $f^m = g^m$ for some $m \geq 2$, f is not conjugate to g , and none of the conjugates of f commutes with g . Furthermore, if the genus of Σ is $\equiv 0 \pmod{4}$, then we can take $m = 2$. Finally, we show that, if Γ is a pure subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ and $f \in \Gamma$, then f has at most one m -root in Γ for all $m \geq 1$. Note that there are finite index pure subgroups in $\mathcal{M}(\Sigma, \mathcal{P})$.

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1 Introduction

Throughout the paper Σ will denote a compact oriented surface, possibly with boundary, and $\mathcal{P} = \{P_1, \dots, P_n\}$ a finite collection of points, called the *punctures*, in the interior of Σ . We denote by $\text{Diff}(\Sigma, \mathcal{P})$ the group of orientation preserving diffeomorphisms $F : \Sigma \rightarrow \Sigma$ which are the identity on a neighborhood of the boundary of Σ and such that $F(\mathcal{P}) = \mathcal{P}$. The *mapping class group* of (Σ, \mathcal{P}) is defined to be the group $\mathcal{M}(\Sigma, \mathcal{P}) = \pi_0(\text{Diff}(\Sigma, \mathcal{P}))$ of isotopy classes of elements of $\text{Diff}(\Sigma, \mathcal{P})$. Note that, in the above definition, one may replace $\text{Diff}(\Sigma, \mathcal{P})$ by $\text{Homeo}(\Sigma, \mathcal{P})$, the group of orientation preserving homeomorphisms $F : \Sigma \rightarrow \Sigma$ which are the identity on $\partial\Sigma$ and such that $F(\mathcal{P}) = \mathcal{P}$ (namely, $\mathcal{M}(\Sigma, \mathcal{P}) = \pi_0(\text{Diff}(\Sigma, \mathcal{P})) = \pi_0(\text{Homeo}(\Sigma, \mathcal{P}))$). Note also that the hypothesis that the elements of $\text{Diff}(\Sigma, \mathcal{P})$ restrict to the identity on a neighborhood of $\partial\Sigma$ (and not only on $\partial\Sigma$) is especially needed when considering subsurfaces. Indeed, if Σ' is a subsurface of Σ such that $\mathcal{P} \cap \partial\Sigma' = \emptyset$, then the embedding $\Sigma' \subset \Sigma$ determines an embedding $\text{Diff}(\Sigma', \mathcal{P} \cap \Sigma') \rightarrow \text{Diff}(\Sigma, \mathcal{P})$ by extending each diffeomorphism $F \in \text{Diff}(\Sigma', \mathcal{P} \cap \Sigma')$ with the identity map outside Σ' , and this monomorphism determines a homomorphism $\mathcal{M}(\Sigma', \mathcal{P} \cap \Sigma') \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$ which is injective in most of the cases but not always (see [29]).

If $\Sigma = \mathbb{D}$ is the standard disk and $|\mathcal{P}| = n$, then $\mathcal{M}(\Sigma, \mathcal{P})$ is the braid group \mathcal{B}_n introduced by Artin [2], [3].

Let Γ be a group, $g \in \Gamma$, and $m \geq 1$. Define a m -root of g to be an element $f \in \Gamma$ such that $f^m = g$. If $\Gamma = \mathbb{Z}^q$, then each element $g \in \Gamma$ has at most one m -root. The same result is true if Γ is free or, more generally, if Γ is biorderable (see Subsection 3.1).

The purpose of the present paper is the study of the roots (or, more precisely, the uniqueness of the roots) in the mapping class groups. Our starting point is a recent result due to González-Meneses [14] which asserts that, in the braid group \mathcal{B}_n , a m -root is unique up to conjugation. This result, also known as the *Makanin conjecture*, has been for some time a classical problem in the field (see [4]).

The braid group has different definitions. One of them is as the mapping class group of the punctured disk, but this is not the only one. So, it is not a surprise that many results on the braid groups (the linearity, for example) are not known for the mapping class groups because their proofs are based on different approaches. But, in what concerns the result of González-Meneses, the proof strongly uses the so-called *Nielsen-Thurston classification* which is one of the main tools in the theory of the mapping class groups.

We briefly recall this classification and refer to Section 2 for detailed definitions and properties. An *essential curve* is defined to be an embedding $a : \mathbb{S}^1 \hookrightarrow \Sigma \setminus \mathcal{P}$ of the standard circle \mathbb{S}^1 which is not parallel to a boundary component of Σ and which does not bound a disk embedded in Σ containing 0 or 1 puncture. The mapping class group $\mathcal{M}(\Sigma, \mathcal{P})$ acts on the set $\mathcal{C}_0(\Sigma, \mathcal{P})$ of isotopy classes of essential curves. We call an element $f \in \mathcal{M}(\Sigma, \mathcal{P})$ *periodic* if some non-trivial power of f acts trivially on $\mathcal{C}_0(\Sigma, \mathcal{P})$, we call f *pseudo-Anosov* if it has no finite orbit in $\mathcal{C}_0(\Sigma, \mathcal{P})$, and we call f *reducible* otherwise. Note that a Dehn twist along a boundary component acts trivially on $\mathcal{C}_0(\Sigma, \mathcal{P})$, so, according to the above definition, it is periodic. In particular, a periodic element does not need to be of finite order. Actually, if $\partial\Sigma \neq \emptyset$, then $\mathcal{M}(\Sigma, \mathcal{P})$ is torsion free (see [30]). When $\partial\Sigma = \emptyset$, the elements of all these three classes (periodic, pseudo-Anosov, reducible) can be represented by diffeomorphisms having some special properties, and these special representatives play a crucial role in the understanding of their corresponding mapping classes.

Now, as pointed out by Birman [6], the result of González-Meneses cannot be extended to the mapping class groups. The most obvious reason is because, if $\partial\Sigma = \emptyset$, then $\mathcal{M}(\Sigma, \mathcal{P})$ has torsion. For example, if $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ are such that f is of order 2 and g is of order 3, then $f^6 = g^6 = \text{Id}$, but f and g are not conjugate. More elementary, if $f \in \mathcal{M}(\Sigma, \mathcal{P})$ is of order $m \geq 2$, then $f^m = \text{Id}^m = \text{Id}$ but f and Id are not conjugate. However, if $\partial\Sigma \neq \emptyset$, the mapping class group $\mathcal{M}(\Sigma, \mathcal{P})$ is torsion free. Besides, our first positive result is the following.

Proposition 3.2. *Let Σ be a surface of genus 0. Then $\mathcal{M}(\Sigma)$ is biorderable. In particular, if $f, g \in \mathcal{M}(\Sigma)$ are such that $f^m = g^m$ for some $m \geq 1$, then $f = g$.*

The case of the surfaces of genus 1 is not so nice, but we still have a positive answer:

Theorem 3.6. *Let Σ be a surface of genus 1 with non-empty boundary. Let $f, g \in \mathcal{M}(\Sigma)$. If $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate.*

The proof of Theorem 3.6 is similar to the one of the theorem of González-Meneses in [14] for the braid groups in the sense that it is based on a deep analysis of the different possible reductions

for a mapping class. However, we also have to determine the conjugacy classes of the periodic elements (and of some special reducible elements), while such a step is not needed in the proof of González-Meneses because the conjugacy classes of the periodic elements in the braid groups were previously determined (see [9], [11], [23]).

Unfortunately, the good news stop at the genus 1. Indeed, we prove the following.

Corollary 4.3. *Let Σ be a surface of genus $\rho \geq 2$ and q boundary components, where $q \geq 1$, and let \mathcal{P} be a finite set of punctures in the interior of Σ . Then there exist $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $f^2 = g^2$, f and g are not conjugate, and none of the conjugates of f commutes with g .*

Our next step is to focus our study on the roots of the pseudo-Anosov elements. In the case where $\partial\Sigma \neq \emptyset$ we prove the following.

Theorem 4.5. *Let Σ be a surface of genus $\rho \geq 0$ and q boundary components, where $q \geq 1$, and let \mathcal{P} be a finite set of punctures in the interior of Σ . Let $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ be two pseudo-Anosov elements. If $f^m = g^m$ for some $m \geq 1$, then $f = g$.*

However, if $\partial\Sigma = \emptyset$, we may have pretty bad examples:

Theorem 5.2.

1. *Let Σ be a closed surface of genus $\rho \geq 2$. Then there exist two pseudo-Anosov elements $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $f^{2(\rho+1)} = g^{2(\rho+1)}$, f is not conjugate to g , and none of the conjugates of f commutes with g .*
2. *Let Σ be a closed surface of genus $\rho \geq 4$, with $\rho \equiv 0 \pmod{4}$. Then there exist two pseudo-Anosov elements $f, g \in \mathcal{M}(\Sigma)$ such that $f^2 = g^2$, f is not conjugate to g , and none of the conjugates of f commutes with g .*

In particular, Theorem 5.2 contradicts the “popular idea” that two pseudo-Anosov elements either commute or generate a free group: the examples of Theorem 5.2 “highly” do not commute and “highly” do not generate a free group. Nevertheless, the set of $g \in \mathcal{M}(\Sigma, \mathcal{P})$ having a common power with a given pseudo-Anosov element f is finite, namely:

Proposition 5.1. *Let Σ be a closed surface of genus $\rho \geq 2$, and let \mathcal{P} be a finite set of punctures in Σ . Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$ be a pseudo-Anosov element. Then the set of $g \in \mathcal{M}(\Sigma, \mathcal{P})$ satisfying $g^m = f^m$ for some $m \geq 1$ is finite.*

Section 6 of the paper is dedicated to the problem of the roots in some special (finite index) subgroups of $\mathcal{M}(\Sigma, \mathcal{P})$. More precisely, we prove the following.

Theorem 6.1. *Let Σ be a surface of genus $\rho \geq 0$ and q boundary components, where $q \geq 0$, and let \mathcal{P} be a finite set of punctures in the interior of Σ . Let Γ be a pure subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$. If $f, g \in \Gamma$ are such that $f^m = g^m$ for some $m \geq 1$, then $f = g$.*

We refer to Section 2 for the definition of a pure subgroup. However, we insist on the fact that f, g must belong to the same pure subgroup in the above statement. The hypothesis that both,

f and g , are pure does not suffice to get the result. For instance, the pseudo-Anosov elements are pure while Theorem 5.2 shows that there exist two different (and, even worse, non-conjugate) pseudo-Anosov elements f, g such that $f^2 = g^2$. The standard example of a pure subgroup is the so-called *Ivanov subgroup* $\Gamma(k)$ which is defined to be the kernel of the natural homomorphism $\mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \text{Aut}(H_1(\Sigma \setminus \mathcal{P}, \mathbb{Z}/k\mathbb{Z}))$, where $k \geq 3$ (see [19]). Note that $\Gamma(k)$ has finite index in $\mathcal{M}(\Sigma, \mathcal{P})$, thus there are finite index subgroups in $\mathcal{M}(\Sigma, \mathcal{P})$ which have the property that each of its elements has at most one m -root inside the subgroup, for all $m \geq 1$. This property is a typical property of the biorderable groups, and it is an open question to know whether $\Gamma(k)$ is biorderable (see [10], [28]).

In order to achieve our main constructions we develop in Section 2 some more or less original techniques which may be interesting by themselves for the reader.

As the reader may notice, we tried to make a treatment of the question of the roots as complete as possible for the mapping class groups of the unpunctured surfaces, and we considered the punctured surfaces only if needed or if the proof do not require any extra argument. So, some problems which essentially concern the mapping class groups of the punctured surfaces are left. Here are some.

Question 1. Let Σ be a surface of genus 0 and q boundary components, where $q \geq 1$, and let \mathcal{P} be a finite set of punctures in the interior of Σ . We suspect that, if $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ are such that $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate. This is true if either $q = 1$ (by [14]) or $\mathcal{P} = \emptyset$ (by Proposition 3.2).

Question 2. Let Σ be a surface of genus 1 with q boundary components, where $q \geq 1$, and let \mathcal{P} be a non-empty set of punctures in the interior of Σ . We do not know whether, if $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ are such that $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate.

Question 3. Let Σ be a closed surface of genus $\rho \geq 2$, and let \mathcal{P} be a non-empty finite set of punctures in Σ . We do not know any necessary and sufficient condition on (Σ, \mathcal{P}) so that there exist two pseudo-Anosov elements $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $f^m = g^m$ for some $m \geq 1$, f and g are not conjugate, and none of the conjugates of f commutes with g . If $|\mathcal{P}| = 1$, then two pseudo-Anosov elements $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ which satisfy $f^m = g^m$ for some $m \geq 1$ must commute. (This can be easily proved using the techniques of the present paper.)

2 Preliminaries

2.1 The complex of curves and the Nielsen-Thurston classification

Let $\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$ denote the standard circle. A *simple closed curve* of (Σ, \mathcal{P}) is an embedding $c : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$. Here we assume that the circle \mathbb{S}^1 as well as the simple closed curves are oriented. The curve with the same image as c but reverse orientation is denoted by c^{-1} (that is, $c^{-1}(z) = c(\bar{z})$ for all $z \in \mathbb{S}^1$). We will also often identify a curve c with its oriented image $c(\mathbb{S}^1)$. The orientation of the surface Σ determines an orientation of each component of $\partial\Sigma$ which, by this way, is viewed as a simple closed curve. Two simple closed curves a and b are *isotopic* if there is a continuous family $\{a_t\}_{t \in [0,1]}$ of simple closed curves such that $a_0 = a$ and $a_1 = b$.

Let $a : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ be a simple closed curve such that $a \cap \partial\Sigma = \emptyset$. A *tubular neighborhood* of a is an embedding $A : [0, 1] \times \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ such that $A(\frac{1}{2}, z) = a(z)$ for all $z \in \mathbb{S}^1$. Choose such a tubular neighborhood and a smooth function $\varphi : [0, 1] \rightarrow \mathbb{R}$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(0) = \varphi'(1) = 0$, and $\varphi'(t) \geq 0$ for all $t \in [0, 1]$, and consider the diffeomorphism $T : \Sigma \rightarrow \Sigma$ which is the identity outside the image of A and which is defined by

$$(T \circ A)(t, z) = A(t, \exp(2i\pi\varphi(t)) \cdot z)$$

inside the image of A . The *Dehn twist* along a is defined to be the isotopy class $\tau_a \in \mathcal{M}(\Sigma, \mathcal{P})$ of $T \in \text{Diff}(\Sigma, \mathcal{P})$. Note that τ_a does not depend neither on the choice of the tubular neighborhood, nor on the choice of the function φ . Moreover, we have $\tau_a = \tau_b$ if a is isotopic to either b or b^{-1} . If a bounds a disk embedded in Σ which contains 0 or 1 puncture, then $\tau_a = 1$. Otherwise, τ_a has infinite order. If $a : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ is a simple closed curve which intersects $\partial\Sigma$ (for example, if a is a boundary component), then the *Dehn twist* along a , denoted by τ_a , is defined to be the Dehn twist along some simple closed curve $b : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ isotopic to a and satisfying $b \cap \partial\Sigma = \emptyset$.

The complex of curves. A simple closed curve $a : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ is called *essential* if neither a , nor a^{-1} , is isotopic to a boundary component of Σ , and if it does not bound a disk embedded in Σ containing 0 or 1 puncture. Two simple closed curves a and b are *equivalent* if either a or a^{-1} is isotopic to b . The equivalence class of a curve a is denoted by $\langle a \rangle$. The vertices of the *complex of curves* $\mathcal{C}(\Sigma, \mathcal{P})$ are the equivalence classes of essential curves. A subset $\{\gamma_0, \gamma_1, \dots, \gamma_p\}$ of vertices is a p -simplex of $\mathcal{C}(\Sigma, \mathcal{P})$ if there exist essential curves c_0, c_1, \dots, c_p such that $\gamma_i = \langle c_i \rangle$ for all $0 \leq i \leq p$, $c_i \cap c_j = \emptyset$ for all $0 \leq i \neq j \leq p$, and c_i is not isotopic to $c_j^{\pm 1}$ (namely, $\gamma_i \neq \gamma_j$) for all $0 \leq i \neq j \leq p$. It is a (non-trivial) fact that $\mathcal{C}(\Sigma, \mathcal{P})$ is a well-defined simplicial complex. Furthermore, the mapping class group $\mathcal{M}(\Sigma, \mathcal{P})$ acts naturally on $\mathcal{C}(\Sigma, \mathcal{P})$, and this action is faithful if $\partial\Sigma = \emptyset$.

The complexes of curves have been introduced by Harvey [17], [18] and are a fundamental tool in the study of the mapping class groups. They are simply connected in general (see [16]) and, if $\partial\Sigma = \emptyset$, then $\text{Aut}(\mathcal{C}(\Sigma, \mathcal{P}))$ coincides with the so-called extended mapping class group (which is defined in the same way as the mapping class group but with diffeomorphisms $F : \Sigma \rightarrow \Sigma$ which do not need to preserve the orientation) (see [20], [25], [27]). They may also be used to defined the so-called *Nielsen-Thurston classification* of the elements of $\mathcal{M}(\Sigma, \mathcal{P})$ as we turn now to do.

The Nielsen-Thurston classification. Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$. We call f *pseudo-Anosov* if it has no finite orbit in $\mathcal{C}(\Sigma, \mathcal{P})$, we call f *periodic* if f^m acts trivially on $\mathcal{C}(\Sigma, \mathcal{P})$ for some $m \geq 1$, and we call f *reducible* otherwise.

As pointed out in the introduction, this classification is of importance in the theory because, when $\partial\Sigma = \emptyset$, then the elements of all these three classes (periodic, pseudo-Anosov, reducible) can be represented by diffeomorphisms having some special properties: a pseudo-Anosov element is represented by a pseudo-Anosov diffeomorphism (see Subsection 2.2), a periodic element is represented by a finite order diffeomorphism (see Subsection 2.8), and a reducible element is represented by a diffeomorphism which globally leaves invariant a family of pairwise disjoint essential curves. Cutting the surface along these curves, this diffeomorphism can be viewed (in some sense) as a diffeomorphism of a “simpler” surface (but not necessarily connected). Some

reducible elements, called the pure elements, will be considered in Subsection 2.7 and in Section 6.

Now, observe that the Dehn twist τ_c along a boundary component c acts trivially on $\mathcal{C}(\Sigma, \mathcal{P})$. In particular, τ_c is periodic. More generally, the kernel of the homomorphism $\mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \text{Aut}(\mathcal{C}(\Sigma, \mathcal{P}))$ has the following geometric interpretation.

Let c_1, \dots, c_q be the boundary components of Σ . Let $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the standard disk. For each $1 \leq i \leq q$ we take a copy \mathbb{D}_i of \mathbb{D} and we denote by $d_i : \mathbb{S}^1 \rightarrow \mathbb{D}_i$ the boundary component of \mathbb{D}_i . Set $\Sigma_0 = (\Sigma \sqcup (\sqcup_{1 \leq i \leq q} \mathbb{D}_i)) / \sim$, where \sim is the equivalence relation which identifies $c_i(z)$ with $d_i(\bar{z}) = d_i^{-1}(z)$ for all $z \in \mathbb{S}^1$. In other words, Σ_0 is the surface obtained from Σ gluing a disk along each boundary component (see Figure 2.1). Choose a point Q_i in the interior of \mathbb{D}_i (say $Q_i = 0$) for all $1 \leq i \leq q$ and set $\mathcal{Q} = \{Q_1, \dots, Q_q\}$. Then the embedding $(\Sigma, \mathcal{P}) \hookrightarrow (\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ determines a homomorphism $\theta : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ that we call the *corking* of (Σ, \mathcal{P}) . Note that θ is not surjective in general. Its image, $\text{Im}\theta$, is the subgroup of $f \in \mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ such that $f(\mathcal{P}) = \mathcal{P}$ and $f(Q_i) = Q_i$ for all $1 \leq i \leq q$. In particular, $\text{Im}\theta$ has finite index in $\mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$.

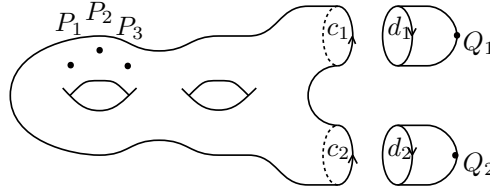


Figure 2.1. A corking.

The following result can be found for instance in [29].

Proposition 2.1. *Assume $|\mathcal{P}| \geq 2$ if $\Sigma = \mathbb{D}$ is the disk and $|\mathcal{P}| \geq 1$ if Σ is the annulus. Let K be the subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ generated by the Dehn twists $\tau_{c_1}, \dots, \tau_{c_q}$ along the boundary components of Σ . Then $K = \text{Ker}\theta$, it is a free abelian group freely generated by $\{\tau_{c_1}, \dots, \tau_{c_q}\}$, and it is contained in the center of $\mathcal{M}(\Sigma, \mathcal{P})$.*

Now, observe that $\mathcal{C}(\Sigma, \mathcal{P}) = \mathcal{C}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ and that every $f \in \mathcal{M}(\Sigma, \mathcal{P})$ acts on $\mathcal{C}(\Sigma, \mathcal{P})$ in the same way as $\theta(f)$. Moreover, $\mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ acts faithfully on $\mathcal{C}(\Sigma, \mathcal{P}) = \mathcal{C}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$, thus the kernel of $\mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \text{Aut}(\mathcal{C}(\Sigma, \mathcal{P}))$ is precisely K . Observe also that f is pseudo-Anosov if and only if $\theta(f)$ is pseudo-Anosov, f is periodic if and only if $\theta(f)$ is periodic, and f is reducible if and only if $\theta(f)$ is reducible.

2.2 Transverse singular foliations and pseudo-Anosov diffeomorphisms

Throughout this subsection the surface Σ is assumed to be closed (namely, $\partial\Sigma = \emptyset$). We shall also assume $|\mathcal{P}| \geq 3$ if Σ is the sphere and $|\mathcal{P}| \geq 1$ if Σ is the torus, so that the Euler characteristic of $\Sigma \setminus \mathcal{P}$ is < 0 . Note that this last restriction is not important since $\mathcal{M}(\mathbb{S}^2) = \mathcal{M}(\mathbb{S}^2, 1) = \mathcal{M}(\mathbb{S}^2, 2) = \{1\}$ and $\mathcal{M}(\mathbb{T}^2) = \mathcal{M}(\mathbb{T}^2, 1)$ (see [5], Chapter 4). The basic reference for the material explained in this subsection is [12]. Other references are [8], [32], and [19].

Transverse singular foliations. A pair of *transverse singular foliations* $(\mathcal{F}_1, \mathcal{F}_2)$ on Σ is a finite set \mathcal{S} of *singularities* together with a covering of $\Sigma \setminus \mathcal{S}$ by an atlas of charts $\{\varphi_i : U_i \rightarrow]-1, 1[^2\}_{i \in I}$ such that the changes of charts preserve the product structure, that is, for all $i, j \in I$ and all $x \in U_i \cap U_j$, there exists an open neighborhood $U \subset U_i \cap U_j$ of x on which the change of coordinates $\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U) \rightarrow \varphi_j(U)$ is of the form $\psi_{ij}(t_1, t_2) = (\psi_{ij}^{(1)}(t_1), \psi_{ij}^{(2)}(t_2))$. The *leaves* of \mathcal{F}_1 (resp. \mathcal{F}_2) are the classes of the smallest equivalence relation \approx_1 (resp. \approx_2) on $\Sigma \setminus \mathcal{S}$ such that $x \approx_1 y$ (resp. $x \approx_2 y$) whenever $x, y \in U_i$ for some $i \in I$, and $\varphi_i(x)$ and $\varphi_i(y)$ belong to the same horizontal (resp. vertical) segment (namely, $\varphi_i(x)$ and $\varphi_i(y)$ have the same second (resp. first) coordinate).

From now on, we shall always assume the singularities to be of the following sort.

Let $P \in \mathcal{S}$ be a singularity and let k be an even positive number (say $k = 2k'$). We say that P is a *singularity with k prongs* (see Figure 2.2) if there exist an open disk U embedded in Σ and centered at P and a k' -folds ramified cover $\pi : U \rightarrow]-1, 1[^2$ such that

- $\pi(P) = (0, 0)$, and $(0, 0)$ is the unique ramification of π ,
- if \mathcal{L}_1 (resp. \mathcal{L}_2) is a leaf of \mathcal{F}_1 (resp. \mathcal{F}_2), then the image of a connected component of $\mathcal{L}_1 \cap U$ (resp. $\mathcal{L}_2 \cap U$) is contained in a horizontal (resp. vertical) segment of $] - 1, 1[^2$.

Note that, if $k = 2$, then π is a regular chart and P may be viewed as a regular point. The *prongs* of \mathcal{F}_1 (resp. \mathcal{F}_2) at P are defined to be the connected components of $\pi^{-1}([[-1, 0[\cup]0, 1] \times \{0\})$ (resp. $\pi^{-1}(\{0\} \times]-1, 0[\cup]0, 1])$).

Let $P \in \mathcal{S}$ be a singularity and let k be an odd positive number. We say that P is a *singularity with k prongs* (see Figure 2.2) if there exists an open disk U embedded in Σ and centered at P such that: if $\tilde{\pi} : \tilde{U} \rightarrow U$ is the 2-folds ramified cover of U with unique ramification at P , $\tilde{\pi}^{-1}(P) = \tilde{P}$, and $\tilde{\mathcal{F}}_1$ (resp. $\tilde{\mathcal{F}}_2$) is the lift of \mathcal{F}_1 (resp. \mathcal{F}_2) in \tilde{U} , then \tilde{P} is a singularity with $2k$ prongs of $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$. The *prongs* of \mathcal{F}_1 (resp. \mathcal{F}_2) at P are defined to be the images under $\tilde{\pi}$ of the prongs of $\tilde{\mathcal{F}}_1$ (resp. $\tilde{\mathcal{F}}_2$) at \tilde{P} .

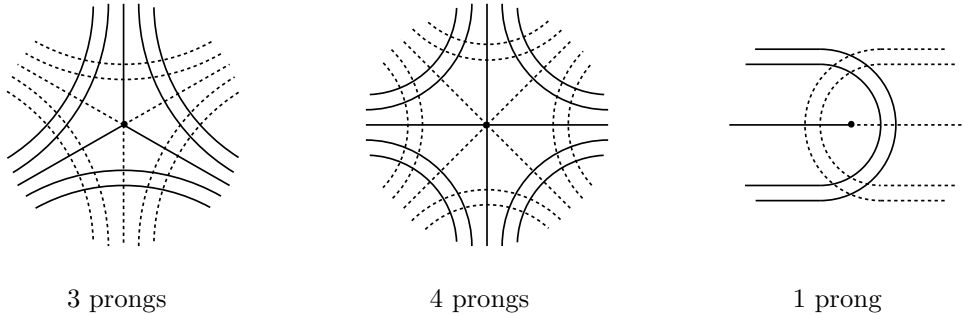


Figure 2.2. Singularities with 1, 3, and 4 prongs.

A leaf of \mathcal{F}_i containing a prong at $P \in \mathcal{S}$ is called a *separatrix* of \mathcal{F}_i at P .

From now on, we always assume the set \mathcal{P} of punctures to be included in \mathcal{S} . However, a puncture $P \in \mathcal{P}$ may be a singularity with 2 prongs.

The *index* of $(\mathcal{F}_1, \mathcal{F}_2)$ at a singularity P is defined to be the number of prongs of \mathcal{F}_i at P and is denoted by $\text{Ind}(\mathcal{F}_1, \mathcal{F}_2 : P)$. The indices and the Euler characteristic of Σ , denoted by $\chi(\Sigma)$, are related by the following formula (see [12], Exposé 5).

Proposition 2.2.

$$\chi(\Sigma) = \sum_{P \in \mathcal{S}} \left(1 - \frac{\text{Ind}(\mathcal{F}_1, \mathcal{F}_2 : P)}{2} \right).$$

□

Transverse invariant measures. A *transverse holonomy invariant measure* μ_1 of \mathcal{F}_1 is a function from the set of (non-necessarily connected) compact curves $\gamma \subset \Sigma$, with values on $[0, +\infty[$, and such that:

- $\mu_1(\gamma_1 \cup \gamma_2) = \mu_1(\gamma_1) + \mu_1(\gamma_2)$ for all compact curves $\gamma_1, \gamma_2 \subset \Sigma$,
- if γ_1, γ_2 are both segments transverse to \mathcal{F}_1 , contained in the source U of a regular chart $\varphi : U \rightarrow]-1, 1]^2$, and having their extremities on the same two horizontal segments of the chart, then $\mu_1(\gamma_1) = \mu_1(\gamma_2)$.

We say that μ_1 has *total support* if $\mu_1(\gamma) > 0$ for every segment $\gamma : [0, 1] \rightarrow \Sigma \setminus \mathcal{S}$ embedded in Σ and (topologically) transverse to \mathcal{F}_1 . We say that the pair of transverse singular foliations $(\mathcal{F}_1, \mathcal{F}_2)$ is *measured* if each \mathcal{F}_i is equipped with a transverse holonomy invariant measure μ_i with total support.

If μ is a transverse holonomy invariant measure of a foliation \mathcal{F} and $\lambda > 0$, then $\lambda\mu$ is also a transverse holonomy invariant measure of \mathcal{F} . Clearly, a transverse holonomy invariant measure is not bounded, so there is no way to normalize it, but one may speak about *classes of measures* which are defined to be the orbits of the transverse measures under the natural action of $]0, +\infty[$. Any foliation of a compact surface admits a transverse holonomy invariant measure (which may be with non-total support). A foliation is called *uniquely ergodic* if it admits exactly one transverse holonomy invariant measure with total support, up to a positive multiplier.

A diffeomorphism $F : \Sigma \rightarrow \Sigma$ acts naturally on the set of pairs of transverse foliations: the pair $(F(\mathcal{F}_1), F(\mathcal{F}_2))$ is defined by the charts $\varphi_i \circ F^{-1} : F(U_i) \rightarrow]-1, 1]^2$, $i \in I$, where $\{\varphi_i : U_i \rightarrow]-1, 1]^2\}_{i \in I}$ is the defining atlas of $(\mathcal{F}_1, \mathcal{F}_2)$. Moreover, if $(\mathcal{F}_1, \mathcal{F}_2)$ is measured, then $(F(\mathcal{F}_1), F(\mathcal{F}_2))$ is also measured: the transverse holonomy invariant measure $F_*(\mu_i)$ is defined by $F_*(\mu_i)(\gamma) = \mu_i(F^{-1}(\gamma))$.

Pseudo-Anosov diffeomorphisms. A diffeomorphism $F \in \text{Diff}(\Sigma, \mathcal{P})$ is called *pseudo-Anosov* if there exist a pair $(\mathcal{F}^s, \mathcal{F}^u)$ of transverse measured singular foliations with measures (μ^s, μ^u) and set of singularities \mathcal{S} , and a real number $\lambda > 1$, such that:

- $F(\mathcal{S}) = \mathcal{S}$, $F(\mathcal{F}^s) = \mathcal{F}^s$, $F(\mathcal{F}^u) = \mathcal{F}^u$, $F_*(\mu^s) = \frac{1}{\lambda}\mu^s$, and $F_*(\mu^u) = \lambda\mu^u$,
- the 1-prongs singularities of $(\mathcal{F}^s, \mathcal{F}^u)$ belong to \mathcal{P} .

We emphasize that a 2-prongs singularity is not a regular point and must be sent to a 2-prongs singularity through a pseudo-Anosov diffeomorphism.

Remark. As pointed out in [12], Exposé 9, a pseudo-Anosov diffeomorphism is in fact a diffeomorphism on $\Sigma \setminus \mathcal{S}$ but is not C^1 at the singularities. However, there are canonical local models for the actions of the pseudo-Anosov diffeomorphisms at the singularities, and this abuse, which is common in the theory, do not alterate the results stated in the remainder.

The following proposition collects some classical results on pseudo-Anosov diffeomorphisms.

Proposition 2.3. *Let $F \in \text{Diff}(\Sigma, \mathcal{P})$ be a pseudo-Anosov diffeomorphism, and let \mathcal{F}^s , \mathcal{F}^u , μ^s , μ^u , and λ as above.*

1. \mathcal{F}^s and \mathcal{F}^u are the unique singular foliations that are invariant under F .
2. (See [12], Exposé 12) \mathcal{F}^s and \mathcal{F}^u are uniquely ergodic.
3. (See [12], Exposé 9) Each separatrix of \mathcal{F}^s (resp. \mathcal{F}^u) contains exactly one prong.
4. (See [12], Exposé 9) Each leaf of either \mathcal{F}^s or \mathcal{F}^u is dense in Σ . More precisely, each half-leaf (connected component of a leaf minus a point) without prong is dense.

□

The above foliations \mathcal{F}^s and \mathcal{F}^u associated to a pseudo-Anosov diffeomorphism F are called, respectively, the *stable* and *unstable foliations* of F . The number $\lambda > 1$ is uniquely determined by F and is called its *dilatation coefficient*.

Now, the coherence of the terminologies introduced in the present subsection and in the previous one is ensured by the following theorem (see [12], Exposé 9).

Theorem 2.4. *The isotopy class of a pseudo-Anosov diffeomorphism is a pseudo-Anosov element of $\mathcal{M}(\Sigma, \mathcal{P})$. Conversely, any pseudo-Anosov element of $\mathcal{M}(\Sigma, \mathcal{P})$ is the isotopy class of a pseudo-Anosov diffeomorphism.*

2.3 Normalizers and symmetries of a pair of measured foliations

We assume again in this subsection that Σ is closed, $|\mathcal{P}| \geq 3$ if Σ is the sphere, and $|\mathcal{P}| \geq 1$ if Σ is the torus.

We fix a pseudo-Anosov diffeomorphism $F \in \text{Diff}(\Sigma, \mathcal{P})$ and we denote by \mathcal{F}^s (resp. \mathcal{F}^u) the stable (resp. unstable) foliation of F , by \mathcal{S} the set of singularities, by (μ^s, μ^u) the measures of $(\mathcal{F}^s, \mathcal{F}^u)$, and by λ the dilatation coefficient of F .

Normalizer of $(\mathcal{F}^s, \mathcal{F}^u)$. Define the *normalizer* of the pair $(\mathcal{F}^s, \mathcal{F}^u)$ to be the group $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ of $G \in \text{Diff}(\Sigma, \mathcal{P})$ such that $G(\mathcal{F}^s) = \mathcal{F}^s$ and $G(\mathcal{F}^u) = \mathcal{F}^u$. Define the *extended normalizer* of $(\mathcal{F}^s, \mathcal{F}^u)$ to be the group $\tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u)$ of $G \in \text{Diff}(\Sigma, \mathcal{P})$ such that $G(\{\mathcal{F}^s, \mathcal{F}^u\}) = \{\mathcal{F}^s, \mathcal{F}^u\}$.

The centralizer of F is contained in $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. Indeed, if G commutes with F , then $G(\mathcal{F}^s)$ and $G(\mathcal{F}^u)$ are, respectively, the stable and unstable foliations of $G \circ F \circ G^{-1} = F$, thus $G(\mathcal{F}^s) = \mathcal{F}^s$ and $G(\mathcal{F}^u) = \mathcal{F}^u$. Note also that either $\tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u) = \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ or $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ is an index 2 subgroup of $\tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u)$.

Let $G \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. Consider the measure $G_*(\mu^u)$ on \mathcal{F}^u . This is a transverse invariant measure on \mathcal{F}^u , thus, by the unique ergodicity of \mathcal{F}^u , there is some number $\lambda^u(G) > 0$ such that $G_*(\mu^u) = \lambda^u(G)\mu^u$. Similarly, there is some number $\lambda^s(G) > 0$ such that $G_*(\mu^s) = \lambda^s(G)\mu^s$. Note also that the map $L : \text{Norm}(\mathcal{F}^s, \mathcal{F}^u) \rightarrow \mathbb{R}$, $G \mapsto \log(\lambda^u(G))$ is a group homomorphism. The proof of the following lemma can be found for instance in [12], Exposé 9.

Lemma 2.5. *Let $G \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. Then $\lambda^s(G) \cdot \lambda^u(G) = 1$.* □

In particular, if $\lambda^u(G) > 1$, then G is a pseudo-Anosov diffeomorphism, \mathcal{F}^s (resp. \mathcal{F}^u) is the stable (resp. unstable) foliation of G , and $\lambda^u(G)$ is its dilatation coefficient. If $\lambda^u(G) < 1$, then G is a pseudo-Anosov diffeomorphism, \mathcal{F}^u (resp. \mathcal{F}^s) is the stable (resp. unstable) foliation of G , and $\lambda^s(G) = \lambda^u(G)^{-1}$ is its dilatation coefficient.

We refer to [1] for the proof of the next lemma.

Lemma 2.6. *The image $L(\text{Norm}(\mathcal{F}^s, \mathcal{F}^u))$ is a discrete subgroup of \mathbb{R} , hence it is of the form $\frac{\log(\lambda)}{k} \cdot \mathbb{Z}$, where $k \in \mathbb{N} \setminus \{0\}$.* □

Symmetries of $(\mathcal{F}^s, \mathcal{F}^u)$. Define the *symmetries* of $(\mathcal{F}^s, \mathcal{F}^u)$ to be the elements of $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u) = \text{Ker}(L)$, and define the *antisymmetries* to be the elements of $\text{Sym}_-(\mathcal{F}^s, \mathcal{F}^u) = \tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u) \setminus \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$.

Note that, being a kernel, $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ is a normal subgroup of $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. More generally, it is easily checked that:

Lemma 2.7. *$\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ is a normal subgroup of $\tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u)$.* □

Now, the coherence of the above terminology is given by the following.

Lemma 2.8. *Let $G \in \text{Sym}_-(\mathcal{F}^s, \mathcal{F}^u)$. Then $G^2 \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$.*

Proof. Let $\lambda_1, \lambda_2 > 0$ be such that $G_*(\mu^s) = \lambda_1\mu^u$ and $G_*(\mu^u) = \lambda_2\mu^s$. Hence $G_*^2(\mu^s) = \lambda_1\lambda_2\mu^s$ and $G_*^2(\mu^u) = \lambda_1\lambda_2\mu^u$, that is, $\lambda^s(G^2) = \lambda^u(G^2) = \lambda_1\lambda_2$. Since $\lambda^s(G^2) \cdot \lambda^u(G^2) = 1$ (see Lemma 2.5), we conclude that $\lambda^s(G^2) = \lambda^u(G^2) = 1$. □

Note that we do not have necessarily $G_1 \circ G_2 \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ if $G_1, G_2 \in \text{Sym}_-(\mathcal{F}^s, \mathcal{F}^u)$, thus $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u) \sqcup \text{Sym}_-(\mathcal{F}^s, \mathcal{F}^u)$ is not a group in general.

Lemma 2.9. *Let G be an element of $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ which fixes some separatrix. Then G is the identity map.*

Proof. Let \mathcal{L} be a stable separatrix of a singular point P which we assume to be fixed under G . In particular, we have $G(P) = P$. Observe that $x \mapsto \int_P^x d\mu^u$ induces a strictly increasing

function from \mathcal{L} (oriented by the origin at P) to $[0, +\infty[$. Since $\lambda^u(G) = 1$, the function G preserves this parametrization of \mathcal{L} , thus the restriction of G to \mathcal{L} is the identity map. We conclude using the fact that \mathcal{L} is dense in Σ (see Proposition 2.3). \square

Corollary 2.10. *The group $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ acts freely on the set of separatrices of \mathcal{F}^s . In particular, $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ is finite.*

Proof. By the Euler-Poincaré formula given in Proposition 2.2, the set \mathcal{S} cannot be empty if Σ is not a torus. On the other hand, we have assumed that $\mathcal{P} \subset \mathcal{S}$, and $\mathcal{P} \neq \emptyset$ if Σ is a torus. So, the set of separatrices is a non-empty finite set and $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ acts freely on it by Lemma 2.9, thus $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ is finite. \square

Lemma 2.11. *Let G be an element of $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ which is isotopic to the identity map. Then G is equal to the identity map.*

Proof. First, observe that $\lambda^s(G) = \lambda^u(G) = 1$ (that is, $G \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$), otherwise G would be a pseudo-Anosov diffeomorphism which cannot be isotopic to the identity by Theorem 2.4. By Corollary 2.10, G has finite order, and, by [12], Exposé 12, any finite order diffeomorphism isotopic to the identity must be the identity map. \square

Corollary 2.12. *The restriction of the natural homomorphism $\text{Diff}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$ to $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ is injective.* \square

2.4 From pseudo-Anosov mapping classes to Pseudo-Anosov diffeomorphisms

We still assume in this section that Σ is a closed surface, that $|\mathcal{P}| \geq 3$ if Σ is the sphere, and that $|\mathcal{P}| \geq 1$ if Σ is the torus. The following theorem (and its corollaries) is the key of many proofs in the present paper. Its proof can be found in [12], Exposé 12 (see also [15]).

Theorem 2.13. *Let $F, G \in \text{Diff}(\Sigma, \mathcal{P})$ be two pseudo-Anosov diffeomorphisms. If F is isotopic to G , then there exists $H \in \text{Diff}(\Sigma, \mathcal{P})$ isotopic to the identity map and such that $G = H \circ F \circ H^{-1}$.* \square

Corollary 2.14. *Let $F \in \text{Diff}(\Sigma, \mathcal{P})$ be a pseudo-Anosov diffeomorphism and $f \in \mathcal{M}(\Sigma, \mathcal{P})$ be its isotopy class. Let $g \in \mathcal{M}(\Sigma, \mathcal{P})$ and $m \geq 1$ such that $f^m = g^m$. Then there is a pseudo-Anosov representative $G \in \text{Diff}(\Sigma, \mathcal{P})$ of g such that $F^m = G^m$.*

Proof. Clearly, the equality $f^m = g^m$ implies that g has no finite orbit in $\mathcal{C}(\Sigma, \mathcal{P})$, that is, g is a pseudo-Anosov element of $\mathcal{M}(\Sigma, \mathcal{P})$. Let $G' \in \text{Diff}(\Sigma, \mathcal{P})$ be a pseudo-Anosov representative of g . The diffeomorphisms F^m and G'^m are both pseudo-Anosov, and they are isotopic, thus, by Theorem 2.13, there exists $H \in \text{Diff}(\Sigma, \mathcal{P})$ isotopic to the identity map such that $H \circ G'^m \circ H^{-1} = F^m$. Set $G = H \circ G' \circ H^{-1}$. Then G is a pseudo-Anosov diffeomorphism, its isotopy class is g , and $G^m = F^m$. \square

Corollary 2.15. *Let $F \in \text{Diff}(\Sigma, \mathcal{P})$ be a pseudo-Anosov diffeomorphism and $f \in \mathcal{M}(\Sigma, \mathcal{P})$ be its isotopy class. Let $g \in \mathcal{M}(\Sigma, \mathcal{P})$ be a pseudo-Anosov element which commutes with f . Then there is a pseudo-Anosov representative $G \in \text{Diff}(\Sigma, \mathcal{P})$ of g such that $F \circ G = G \circ F$.*

Proof. Let $G' \in \text{Diff}(\Sigma, \mathcal{P})$ be a pseudo-Anosov representative of g . The diffeomorphism $G' \circ F \circ G'^{-1}$ is a pseudo-Anosov diffeomorphism isotopic to F , thus, by Theorem 2.13, there exists $H \in \text{Diff}(\Sigma, \mathcal{P})$ isotopic to the identity map such that $H \circ G' \circ F \circ G'^{-1} \circ H^{-1} = F$. Set $G = H \circ G'$. Then $G \circ F = F \circ G$ and G is isotopic to G' , namely, G is a representative of g . So, it remains to prove that G is a pseudo-Anosov diffeomorphism.

Let \mathcal{F}^s (resp. \mathcal{F}^u) be the stable (resp. unstable) foliation of F . As observed before, the fact that G commutes with F implies that it belongs to $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. Moreover, the isotopy class of G is g which is assumed to be pseudo-Anosov, thus $\lambda^s(G) \neq 1$. So, G is a pseudo-Anosov diffeomorphism. \square

2.5 Coverings and pseudo-Anosov diffeomorphisms

We turn now in this subsection to explain a way for constructing pseudo-Anosov diffeomorphisms having precise groups of symmetries by means of ramified coverings. Then the proof of Theorem 5.2 will be a direct application of the present subsection together with the next one.

Lemma 2.16. *Let M be a compact connected manifold, let $\pi : N \rightarrow M$ be a finite covering, possibly with a finite set \mathcal{R} of ramification points, and let $F : M \rightarrow M$ be a homeomorphism such that $F(\mathcal{R}) = \mathcal{R}$. Then there exists $k \geq 1$ such that F^k has a lift on N .*

Proof. We choose a base point $P_0 \in M \setminus \mathcal{R}$ and a path $\gamma : [0, 1] \rightarrow M \setminus \mathcal{R}$ joining P_0 to $F(P_0)$, and we denote by $\phi_\gamma : \pi_1(M \setminus \mathcal{R}, F(P_0)) \rightarrow \pi_1(M \setminus \mathcal{R}, P_0)$ the isomorphism defined by $\alpha \mapsto \gamma^{-1} \alpha \gamma$. The general theory of coverings asserts that $F|_{M \setminus \mathcal{R}}$ has a lift on $N \setminus \pi^{-1}(\mathcal{R})$ if and only if

$$(\phi_\gamma \circ F_*)(\pi_1(N \setminus \pi^{-1}(\mathcal{R}), Q_0)) = \pi_1(N \setminus \pi^{-1}(\mathcal{R}), Q_0),$$

where Q_0 is a chosen element of the preimage of P_0 , and $\pi_1(N \setminus \pi^{-1}(\mathcal{R}), Q_0)$ is viewed as a subgroup of $\pi_1(M \setminus \mathcal{R}, P_0)$. Let m be the number of sheets of the covering. The group $\pi_1(N \setminus \pi^{-1}(\mathcal{R}), Q_0)$ is an index m subgroup of $\pi_1(M \setminus \mathcal{R}, P_0)$ and there are finitely many index m subgroups in $\pi_1(M \setminus \mathcal{R}, P_0)$. Thus, there exists some $k \geq 1$ such that

$$(\phi_\gamma \circ F_*)^k(\pi_1(N \setminus \pi^{-1}(\mathcal{R}), Q_0)) = \pi_1(N \setminus \pi^{-1}(\mathcal{R}), Q_0).$$

On the other hand, we have $(\phi_\gamma \circ F_*)^k = \phi_{\gamma^{(k)}} \circ (F^k)_*$, where

$$\gamma^{(k)} = (F^{k-1} \circ \gamma) \cdot \dots \cdot (F \circ \gamma) \cdot \gamma,$$

which is a path from P_0 to $F^k(P_0)$. So, $F^k|_{M \setminus \mathcal{R}}$ has a lift $\tilde{F} : N \setminus \pi^{-1}(\mathcal{R}) \rightarrow N \setminus \pi^{-1}(\mathcal{R})$. Clearly, \tilde{F} can be extended to a homeomorphism $\tilde{F} : N \rightarrow N$, and \tilde{F} is a lift of $F^k : M \rightarrow M$. \square

We consider a regular ramified covering $\pi : \Sigma \rightarrow \Sigma_0$ of a closed surface Σ_0 . That is, there is a finite group Γ , called the *Galois group* of π , acting on Σ , and such that $\Sigma_0 = \Sigma/\Gamma$ and $\pi : \Sigma \rightarrow \Sigma_0$ is the natural quotient. Let \mathcal{R}_0 be the set of ramification points of π . For each $Q \in \mathcal{R}_0$, we denote by $r_\pi(Q)$ the *ramification order* of π at Q_0 . If $o(Q) = |\pi^{-1}(Q)|$, then this number is defined by $r_\pi(Q) = \frac{m}{o(Q)}$, where m is the number of sheets of π . Here we prescribe that $r_\pi(Q) \geq 2$ for all $Q \in \mathcal{R}_0$, so that there is no regular point hidden in \mathcal{R}_0 . If Q is regular, that is, if $Q \in \Sigma_0 \setminus \mathcal{P}_0$, then the *ramification order* of π at Q is $r_\pi(Q) = 1$.

Let $F_0 \in \text{Diff}(\Sigma_0, \mathcal{R}_0)$ be a pseudo-Anosov diffeomorphism, let \mathcal{F}_0^s (resp. \mathcal{F}_0^u) be the stable (resp. unstable) foliation of F_0 , and let λ be its dilatation coefficient. Then the following lemma can be easily proved with standard arguments.

Lemma 2.17. *The foliation \mathcal{F}_0^s (resp. \mathcal{F}_0^u) lifts to a foliation \mathcal{F}^s (resp. \mathcal{F}^u) on Σ , and $(\mathcal{F}^s, \mathcal{F}^u)$ is a pair of transverse measured singular foliations of Σ . Moreover, we have*

$$\text{Ind}(\mathcal{F}^s, \mathcal{F}^u : P) = \text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \pi(P)) \cdot r_\pi(\pi(P))$$

for all $P \in \Sigma$, and Γ is a subgroup of $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$. \square

We keep the notations and we assume, furthermore, that F_0 lifts to a diffeomorphism $F : \Sigma \rightarrow \Sigma$. (By Lemma 2.16, this is always possible up to some power.) Then F is a pseudo-Anosov diffeomorphism, \mathcal{F}^s (resp. \mathcal{F}^u) is its stable (resp. unstable) foliation, and λ is its dilatation coefficient.

Pivot. A point $Q_0 \in \mathcal{R}_0$ is called a *pivot* of (F_0, π) if it is a singular point with one prong (i.e. $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : Q_0) = 1$), and if $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : Q) \cdot r_\pi(Q) \neq r_\pi(Q_0)$ for all $Q \in \Sigma_0 \setminus \{Q_0\}$.

Proposition 2.18. *If (F_0, π) has a pivot, then $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ is equal to the Galois group Γ .*

Proof. We already know that $\Gamma \subset \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$ (see Lemma 2.17). For $k \geq 3$, we denote by \mathcal{S}_k the set of singularities of $(\mathcal{F}^s, \mathcal{F}^u)$ with k prongs, and by \mathcal{SS}_k the set of separatrices of \mathcal{F}^s at the elements of \mathcal{S}_k . (So, by Proposition 2.3, we have $|\mathcal{SS}_k| = k \cdot |\mathcal{S}_k|$.) Take a pivot $Q_0 \in \mathcal{R}_0$, and set $k_0 = r_\pi(Q_0)$. By Lemma 2.17, the fact that Q_0 is a pivot implies that $\mathcal{S}_{k_0} = \pi^{-1}(Q_0)$. Moreover, the group Γ acts transitively on \mathcal{SS}_{k_0} because the elements of \mathcal{SS}_{k_0} are precisely the preimages under π of the unique separatrix of \mathcal{F}_0^s at Q_0 . We conclude by Lemma 2.9 that $\Gamma = \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$. \square

2.6 Pseudo-Anosov diffeomorphisms which fix their separatrices

Throughout this subsection Σ is a closed surface and $F : \Sigma \rightarrow \Sigma$ is a pseudo-Anosov diffeomorphism. We denote by \mathcal{F}^s (resp. \mathcal{F}^u) the stable (resp. unstable) foliation of F .

Fixed separatrices. We say that F *fixes its separatrices* if it acts trivially on the set of separatrices of \mathcal{F}^s or, equivalently, if it acts trivially on the set of separatrices of \mathcal{F}^u .

Lemma 2.19. *Suppose that F fixes its separatrices.*

1. *The centralizer of F in $\text{Diff}(\Sigma)$ is equal to $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$.*
2. *Let $G \in \tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u) \setminus \text{Norm}(\mathcal{F}^s, \mathcal{F}^u) = \text{Sym}_-(\mathcal{F}^s, \mathcal{F}^u)$. Then $G \circ F \circ G = F^{-1}$.*

Proof. As pointed out in Subsection 2.3, we already know that the centralizer of F is contained in $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. So, it remains to show that $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ is contained in the centralizer of F . Let $G \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. Then $G \circ F \circ G^{-1}$ is a pseudo-Anosov diffeomorphism, its stable (resp. unstable) foliation is \mathcal{F}^s (resp. \mathcal{F}^u), its dilatation coefficient is λ , and it acts trivially on

the set of separatrices of \mathcal{F}^s . We conclude by Lemma 2.9 that $F^{-1} \circ G \circ F \circ G^{-1}$ is the identity map, thus $G \circ F \circ G^{-1} = F$.

Let $G \in \text{Sym}_-(\mathcal{F}^s, \mathcal{F}^u)$. Then $G \circ F \circ G^{-1}$ is a pseudo-Anosov diffeomorphism, its stable (resp. unstable) foliation is \mathcal{F}^u (resp. \mathcal{F}^s), its dilatation coefficient is λ , and it acts trivially on the set of separatrices of \mathcal{F}^s . We conclude by Lemma 2.9 that $F \circ G \circ F \circ G^{-1}$ is the identity map, thus $G \circ F \circ G^{-1} = F^{-1}$. \square

Proposition 2.20. *Suppose that F fixes its separatrices. Let $R_1, R_2 \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$. Write $F_1 = F \circ R_1$ (resp. $F_2 = F \circ R_2$) and denote by $f_1 \in \mathcal{M}(\Sigma)$ (resp. $f_2 \in \mathcal{M}(\Sigma)$) the isotopy class of F_1 (resp. F_2). There exists $h \in \mathcal{M}(\Sigma)$ such that $hf_1h^{-1} = f_2$ if and only if there exists $H \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ R_1 \circ H^{-1} = R_2$.*

Proof. Assume that there exists $H \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ R_1 \circ H^{-1} = R_2$. By Lemma 2.19, we have $H \circ F \circ H^{-1} = F$, thus $H \circ F_1 \circ H^{-1} = F \circ (H \circ R_1 \circ H^{-1}) = F \circ R_2 = F_2$, therefore $hf_1h^{-1} = f_2$, where $h \in \mathcal{M}(\Sigma)$ is the isotopy class of H .

Now, assume that there exists $h \in \mathcal{M}(\Sigma)$ such that $hf_1h^{-1} = f_2$. Let $H_1 \in \text{Diff}(\Sigma)$ which represents h . The diffeomorphisms $H_1 \circ F_1 \circ H_1^{-1}$ and F_2 are both pseudo-Anosov and contained in the same isotopy class, thus, by Theorem 2.13, there exists $H_2 \in \text{Diff}(\Sigma)$ isotopic to the identity map such that $H_2 \circ H_1 \circ F_1 \circ H_1^{-1} \circ H_2^{-1} = F_2$. Set $H = H_2 \circ H_1$. Then $F_2 = H \circ F_1 \circ H^{-1}$.

The foliation \mathcal{F}^s (resp. \mathcal{F}^u) is the stable (resp. unstable) foliation of F_2 , and $H(\mathcal{F}^s)$ (resp. $H(\mathcal{F}^u)$) is the stable (resp. unstable) foliation of $H \circ F_1 \circ H^{-1} = F_2$, thus $H(\mathcal{F}^s) = \mathcal{F}^s$ and $H(\mathcal{F}^u) = \mathcal{F}^u$, that is, $H \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. By Lemma 2.19, we have $H \circ F \circ H^{-1} = F$, thus $F \circ (H \circ R_1 \circ H^{-1}) = H \circ F_1 \circ H^{-1} = F_2 = F \circ R_2$, therefore $H \circ R_1 \circ H^{-1} = R_2$. \square

Proposition 2.21. *Suppose that F fixes its separatrices. Let $R_1, R_2 \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$. Write $F_1 = F \circ R_1$ (resp. $F_2 = F \circ R_2$) and denote by $f_1 \in \mathcal{M}(\Sigma)$ (resp. $f_2 \in \mathcal{M}(\Sigma)$) the isotopy class of F_1 (resp. F_2). There exists $h \in \mathcal{M}(\Sigma)$ such that hf_1h^{-1} commutes with f_2 if and only if there exists $H \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ R_1 \circ H^{-1}$ commutes with R_2 .*

Proof. Assume that there exists $H \in \widetilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ R_1 \circ H^{-1}$ and R_2 commute. By Lemma 2.19, there exists $\varepsilon \in \{\pm 1\}$ such that $H \circ F \circ H^{-1} = F^\varepsilon$. Moreover, again by Lemma 2.19, $H \circ R_1 \circ H^{-1}$ and R_2 commute with F (since they belong to $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u) \subset \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$). We conclude that $H \circ F_1 \circ H^{-1} = F^\varepsilon \circ (H \circ R_1 \circ H^{-1})$ commutes with $F_2 = F \circ R_2$, thus hf_1h^{-1} commutes with f_2 , where $h \in \mathcal{M}(\Sigma)$ is the isotopy class of H .

Now, assume that there exists $h \in \mathcal{M}(\Sigma)$ such that hf_1h^{-1} and f_2 commute. Let $H_1 \in \text{Diff}(\Sigma)$ which represents h . By Corollary 2.15, there exists a pseudo-Anosov diffeomorphism $F'_1 \in \text{Diff}(\Sigma)$ which represents hf_1h^{-1} and which commutes with F_2 . The diffeomorphisms $H_1 \circ F_1 \circ H_1^{-1}$ and F'_1 are both pseudo-Anosov and contained in the same isotopy class, thus, by Theorem 2.13, there exists $H_2 \in \text{Diff}(\Sigma)$ isotopic to the identity map such that $F'_1 = H_2 \circ H_1 \circ F_1 \circ H_1^{-1} \circ H_2^{-1}$. Set $H = H_2 \circ H_1$. Then $H \circ F_1 \circ H^{-1}$ and F_2 commute.

The foliation \mathcal{F}^s (resp. \mathcal{F}^u) is the stable (resp. unstable) foliation of F_2 , and $H \circ F_1 \circ H^{-1}$ belongs to the centralizer of F_2 , thus $(H \circ F_1 \circ H^{-1}) \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$. On the other hand,

we have $(H \circ F_1 \circ H^{-1}) \in \text{Norm}(H(\mathcal{F}^s), H(\mathcal{F}^u))$, because \mathcal{F}^s (resp. \mathcal{F}^u) is the stable (resp. unstable) foliation of F_1 . By Proposition 2.3, it follows that $\{H(\mathcal{F}^s), H(\mathcal{F}^u)\} = \{\mathcal{F}^s, \mathcal{F}^u\}$, thus $H \in \tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u)$. By Lemma 2.19, there exists $\varepsilon \in \{\pm 1\}$ such that $H \circ F \circ H^{-1} = F^\varepsilon$. Moreover, $H \circ R_1 \circ H^{-1}$ and R_2 commute with F (by Lemma 2.19). Since $H \circ F_1 \circ H^{-1} = F^\varepsilon \circ (H \circ R_1 \circ H^{-1})$ commutes with $F_2 = F \circ R_2$, we conclude that $H \circ R_1 \circ H^{-1}$ commutes with R_2 . \square

2.7 Canonical reduction systems and pure elements

Now, the surface Σ is not assumed to be closed anymore. To each reducible element $f \in \mathcal{M}(\Sigma, \mathcal{P})$ one can associate a well-defined simplex $\Delta(f)$ of $\mathcal{C}(\Sigma, \mathcal{P})$ called the *canonical reduction system* for f . It happens that it is easier to first define $\Delta(f)$ for a certain kind of elements, called the *pure elements*, as we turn now to do. On the other hand, the pure subgroups are the main object of Section 6 and here is a good place to introduce them. A standard reference for the material explained in this subsection is [19].

Pure elements and pure subgroups. An element $f \in \mathcal{M}(\Sigma, \mathcal{P})$ is called *pure* if, for all essential curves $c : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$, either $f(c)$ is isotopic to c , or $f^m(c)$ is not isotopic to c for any $m \in \mathbb{Z} \setminus \{0\}$. We emphasize that, by $f(c)$ is isotopic to c , we also mean that f preserves the orientation of c up to isotopy. Note that a pure element has no finite orbit of order ≥ 2 in $\mathcal{C}(\Sigma, \mathcal{P})$, and, if it is not the identity, it has infinite order. The pseudo-Anosov elements and the Dehn twists are examples of pure elements. A subgroup $\Gamma \subset \mathcal{M}(\Sigma, \mathcal{P})$ is called *pure* if all its elements are pure. Note that a pure subgroup is torsion-free.

Example. Let $m \geq 3$, and let $\Gamma(m)$ denote the kernel of the natural homomorphism $\mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \text{Aut}(H_1(\Sigma \setminus \mathcal{P}, \mathbb{Z}/m\mathbb{Z}))$. Then, by [19], $\Gamma(m)$ is a pure subgroup. Note that $\Gamma(m)$ has finite index in $\mathcal{M}(\Sigma, \mathcal{P})$ since $H_1(\Sigma \setminus \mathcal{P}, \mathbb{Z}/m\mathbb{Z})$ is finite. So, there are finite index pure subgroups and, in particular, there are finite index torsion-free subgroups in $\mathcal{M}(\Sigma, \mathcal{P})$.

Let $\Delta = \{\delta_0, \delta_1, \dots, \delta_p\}$ be a simplex of $\mathcal{C}(\Sigma, \mathcal{P})$. We choose essential curves d_0, d_1, \dots, d_p such that $\langle d_i \rangle = \delta_i$ for all $0 \leq i \leq p$, and $d_i \cap d_j = \emptyset$ for all $0 \leq i \neq j \leq p$. We denote by Σ_Δ the surface obtained from Σ cutting along the d_i 's. Let c_1, \dots, c_q be the boundary components of Σ . Then Σ_Δ is a non-necessarily connected compact surface, whose boundary components are $c_1, \dots, c_q, d_0^{(1)}, d_1^{(1)}, \dots, d_p^{(1)}, d_0^{(2)}, d_1^{(2)}, \dots, d_p^{(2)}$, and $\Sigma = \Sigma_\Delta / \sim$, where \sim is the equivalence relation which identifies $d_i^{(1)}(z)$ with $d_i^{(2)}(\bar{z}) = (d_i^{(2)})^{-1}(z)$ for all $0 \leq i \leq p$ and all $z \in \mathbb{S}^1$. Let $\pi_\Delta : \Sigma_\Delta \rightarrow \Sigma$ denote the natural quotient. Then $d_i(z) = \pi_\Delta \circ d_i^{(1)}(z) = \pi_\Delta \circ (d_i^{(2)})^{-1}(z)$ for all $0 \leq i \leq p$ and all $z \in \mathbb{S}^1$. Let $\Sigma_{\Delta 1}, \dots, \Sigma_{\Delta l}$ be the connected components of Σ_Δ , let $\mathcal{P}_\Delta = \pi_\Delta^{-1}(\mathcal{P})$, and let $\mathcal{P}_{\Delta k} = \mathcal{P}_\Delta \cap \Sigma_{\Delta k}$ for all $1 \leq k \leq l$. Set

$$\mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta) = \mathcal{M}(\Sigma_{\Delta 1}, \mathcal{P}_{\Delta 1}) \times \dots \times \mathcal{M}(\Sigma_{\Delta l}, \mathcal{P}_{\Delta l}).$$

Then the map $\pi_\Delta : \Sigma_\Delta \rightarrow \Sigma$ induces a homomorphism $\theta_\Delta : \mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$.

The following result is well-known and its proof is implicit in [29].

Lemma 2.22. For $0 \leq i \leq p$, set $\varphi_i = \tau_{d_i^{(1)}} \cdot \tau_{d_i^{(2)}}^{-1}$. Let K_Δ be the subgroup of $\mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta)$ generated by $\varphi_0, \varphi_1, \dots, \varphi_p$. Then $K_\Delta = \text{Ker} \theta_\Delta$, it is a free abelian group freely generated by $\{\varphi_0, \varphi_1, \dots, \varphi_p\}$, and it is contained in the center of $\mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta)$. \square

Let \tilde{K} be the subgroup of $\mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta)$ generated by $\{\tau_{c_1}, \dots, \tau_{c_q}, \tau_{d_0^{(1)}}, \tau_{d_1^{(1)}}, \dots, \tau_{d_p^{(1)}}, \tau_{d_0^{(2)}}, \tau_{d_1^{(2)}}, \dots, \tau_{d_p^{(2)}}\}$, and, for $1 \leq j \leq l$, let $\tilde{K}_j = \tilde{K} \cap \mathcal{M}(\Sigma_{\Delta_j}, \mathcal{P}_{\Delta_j})$. Then \tilde{K}_j is the subgroup of $\mathcal{M}(\Sigma_{\Delta_j}, \mathcal{P}_{\Delta_j})$ generated by the Dehn twists along the boundary components of Σ_{Δ_j} , and we have $\tilde{K} = \tilde{K}_1 \times \dots \times \tilde{K}_l$.

Define the *stabilizer* of Δ to be $\text{Stab}(\Delta) = \{f \in \mathcal{M}(\Sigma, \mathcal{P}); f(\Delta) = \Delta\}$. The image of θ_Δ is contained in $\text{Stab}(\Delta)$, but, in general, is not equal to $\text{Stab}(\Delta)$. However, if $f \in \text{Stab}(\Delta)$ is pure, then f fixes each d_i up to isotopy, thus it belongs to $\text{Im}(\theta_\Delta)$. Moreover, if $(f_1, \dots, f_l) \in \theta_\Delta^{-1}(f)$, then each f_j is a pure element of $\mathcal{M}(\Sigma_{\Delta_j}, \mathcal{P}_{\Delta_j})$.

Reduction systems. Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$ be a pure element. A simplex Δ of $\mathcal{C}(\Sigma, \mathcal{P})$ is called a *reduction system* for f if $f(\Delta) = \Delta$ (that is, if $f \in \text{Stab}(\Delta)$). Let $(f_1, \dots, f_l) \in \theta_\Delta^{-1}(f)$. If, moreover, for all $1 \leq j \leq l$, either f_j is a pseudo-Anosov element of $\mathcal{M}(\Sigma_{\Delta_j}, \mathcal{P}_{\Delta_j})$, or f_j is an element of \tilde{K}_j , then the reduction system Δ is called a *complete reduction system* for f . If f is reducible, then the intersection of all the complete reduction systems is a complete reduction system itself, called the *canonical reduction system* for f and denoted by $\Delta(f)$. If f is a non-reducible pure element (for example, if f is either a pseudo-Anosov element or a Dehn twist along a boundary component of Σ), then the *canonical reduction system* for f is defined to be $\Delta(f) = \emptyset$. If $f \in \mathcal{M}(\Sigma, \mathcal{P})$ is a pure element and $m \in \mathbb{Z} \setminus \{0\}$, then f^m is also pure and $\Delta(f^m) = \Delta(f)$. In particular, if f and g are both pure elements and $f^m = g^m$ for some $m \in \mathbb{Z} \setminus \{0\}$, then $\Delta(f) = \Delta(g)$.

The notion of a canonical reduction system can be extended to all the elements of $\mathcal{M}(\Sigma, \mathcal{P})$ using the following result which can be found for instance in [19] and [7].

Proposition 2.23. Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$ be a reducible element. Then there exists a unique simplex $\Delta = \Delta(f)$ of $\mathcal{C}(\Sigma, \mathcal{P})$ such that:

- $f(\Delta) = \Delta$;
- there exists $m \geq 1$ such that f^m is pure and $\Delta = \Delta(f^m)$.

\square

Canonical reduction systems. The above simplex $\Delta(f)$ is called the *canonical reduction system* for f , and, if f is non-reducible (namely, is either pseudo-Anosov or periodic), then the *canonical reduction system* for f is defined to be $\Delta(f) = \emptyset$.

2.8 The Kerckhoff realization theorem

Throughout this subsection the surface Σ is again assumed to be closed, $|\mathcal{P}| \geq 3$ if Σ is the sphere, and $|\mathcal{P}| \geq 1$ if Σ is the torus.

Theorem 2.24 (Kerckhoff [22]). *Assume $\mathcal{P} = \emptyset$. Let Γ be a finite subgroup of $\mathcal{M}(\Sigma)$. Then there exist a metric g on Σ with constant curvature -1 and a finite subgroup $\tilde{\Gamma} \subset \text{Isom}(\Sigma, g)$ such that $\tilde{\Gamma}$ is sent isomorphically onto Γ under the natural homomorphism $\text{Isom}(\Sigma, g) \rightarrow \mathcal{M}(\Sigma)$. \square*

The above result, known as the Kerckhoff realization theorem, can be extended to the mapping class groups of the punctured surfaces as follows.

Theorem 2.25. *Let Γ be a finite subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$. Then there exists a finite subgroup $\tilde{\Gamma} \subset \text{Diff}(\Sigma, \mathcal{P})$ which is sent isomorphically onto Γ under the natural homomorphism $\text{Diff}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$.*

A proof of Theorem 2.25 which extends Kerckhoff's arguments is sketched in [21]. However, one can also use Theorem 2.24 together with the following construction (which may be useful for other purposes) in order to achieve the result.

Write $\mathcal{P} = \{P_1, \dots, P_n\}$. For each $i \in \{1, \dots, n\}$, we choose a small disk \mathbb{D}_i embedded in Σ and containing P_i in its interior. Moreover, we choose the \mathbb{D}_i 's so that $\mathbb{D}_i \cap \mathbb{D}_j = \emptyset$ for all $1 \leq i \neq j \leq n$. We denote by $\overset{\circ}{\mathbb{D}}_i$ the interior of \mathbb{D}_i and by $c_i : \mathbb{S}^1 \rightarrow \mathbb{D}_i$ its boundary component. Set $\Sigma_1 = \Sigma \setminus (\cup_{i=1}^n \overset{\circ}{\mathbb{D}}_i)$. Then Σ_1 is a compact surface whose boundary components are $c_1^{-1}, \dots, c_n^{-1}$. Now, take a copy Σ'_1 of Σ_1 and define the closed surface $\hat{\Sigma} = (\Sigma_1 \sqcup \Sigma'_1) / \sim$, where \sim is the equivalence relation which identifies $c_i(z) \in \partial\Sigma_1$ with $c_i(\bar{z}) = c_i^{-1}(z) \in \partial\Sigma'_1$ for all $1 \leq i \leq n$ and all $z \in \mathbb{S}^1$ (see Figure 2.3).

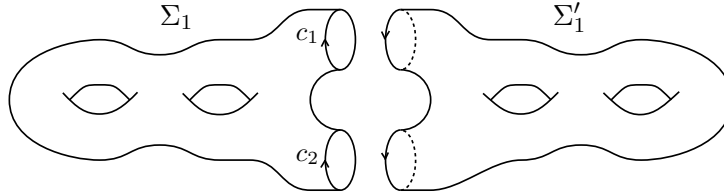


Figure 2.3. The surface $\hat{\Sigma}$.

We take a diffeomorphism $\eta : \Sigma_1 \rightarrow \Sigma_1$ of order 2, which reverses the orientation, and such that $(\eta \circ c_i)(z) = c_i(\bar{z})$ for all $1 \leq i \leq n$ and all $z \in \mathbb{S}^1$. Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$, and let $\sigma \in \text{Sym}_n$ such that $f(P_i) = P_{\sigma(i)}$ for all $1 \leq i \leq n$. We choose a representative $F : \Sigma \rightarrow \Sigma$ of f such that $F \circ c_i = c_{\sigma(i)}$ and $F(\mathbb{D}_i) = \mathbb{D}_{\sigma(i)}$ for all $1 \leq i \leq n$. Then F restricts to a diffeomorphism $F : \Sigma_1 \rightarrow \Sigma_1$ (which does not necessarily belong to $\text{Diff}(\Sigma_1)$ since it does not need to be the identity on $\partial\Sigma_1$). Let $\hat{F} : \hat{\Sigma} \rightarrow \hat{\Sigma}$ be defined by $\hat{F}|_{\Sigma_1} = F$ and $\hat{F}|_{\Sigma'_1} = \eta \circ F \circ \eta$. Then \hat{F} is a well-defined homeomorphism, and it is a diffeomorphism on $\hat{\Sigma} \setminus (\cup_{i=1}^n c_i)$. Moreover, we can and do choose F such that \hat{F} is a diffeomorphism on the whole surface $\hat{\Sigma}$.

The proof of the following lemma is left to the reader.

Lemma 2.26. *The mapping $\mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \text{Diff}(\hat{\Sigma})$, $f \mapsto \hat{F}$ determines a well-defined injective homomorphism $\iota : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\hat{\Sigma})$. \square*

Proof of Theorem 2.25. Let Γ be a finite subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$. Consider the image $\iota(\Gamma)$ of Γ under the monomorphism $\iota : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\hat{\Sigma})$. By Theorem 2.24, there exist a metric \hat{g} on $\hat{\Sigma}$ of constant curvature -1 and a finite subgroup $\hat{\Gamma}$ of $\text{Isom}(\hat{\Sigma}, \hat{g})$ such that $\hat{\Gamma}$ is sent isomorphically onto $\iota(\Gamma)$ under the natural homomorphism $\text{Isom}(\hat{\Sigma}, \hat{g}) \rightarrow \mathcal{M}(\hat{\Sigma})$. We can and do assume that each c_i is the unique geodesic circle in its isotopy class. Let $\hat{F} \in \hat{\Gamma}$. Then $\hat{F}(\Sigma_1) = \Sigma_1$, $\hat{F}(\Sigma'_1) = \Sigma'_1$, and there exists a permutation $\sigma \in \text{Sym}_n$ such that $\hat{F}(c_i) = c_{\sigma(i)}$ for all $1 \leq i \leq n$. We endow each \mathbb{D}_i with the flat metric of radius $\text{length}(c_i)/2\pi$ and assume that P_i is the center of \mathbb{D}_i . Then $\hat{F}|_{\Sigma_1}$ extends in a unique way to an isometry $F : \Sigma \rightarrow \Sigma$ which satisfies $F(\mathcal{P}) = \mathcal{P}$. Now, the mapping $\hat{\Gamma} \rightarrow \text{Isom}(\Sigma, \mathcal{P}) \subset \text{Diff}(\Sigma, \mathcal{P})$, $\hat{F} \mapsto F$, is a well-defined homomorphism. Its image, $\tilde{\Gamma}$, is a finite subgroup of $\text{Diff}(\Sigma, \mathcal{P})$ which is sent isomorphically onto Γ under the natural homomorphism $\text{Diff}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$. \square

We finish this subsection with a simple and well-known result whose proof is left to the reader.

Lemma 2.27. *Let Σ be a closed surface, and let $\Gamma \subset \text{Diff}(\Sigma)$ be a finite subgroup of order $m \geq 1$. Then Σ/Γ is a surface and the natural quotient $\pi : \Sigma \rightarrow \Sigma/\Gamma$ is a ramified covering. Let Q_1, \dots, Q_l be the ramification points of π , and, for $1 \leq i \leq l$, let $o(Q_i) = |\pi^{-1}(Q_i)|$ be the cardinality of $\pi^{-1}(Q_i)$. Then the Euler characteristics of Σ and Σ/Γ are related by the formula*

$$\chi(\Sigma) + \sum_{i=1}^l (m - o(Q_i)) = m \cdot \chi(\Sigma/\Gamma).$$

\square

3 Low genus surfaces

3.1 Genus 0 surfaces

A group Γ is *biorderable* if there exists a linear ordering $<$ on Γ invariant by left and right multiplications (namely, if $f < g$, then $h_1 f h_2 < h_1 g h_2$, for all $f, g, h_1, h_2 \in \Gamma$).

Example. Let $\pi : \mathcal{B}_n \rightarrow \text{Sym}_n$ be the natural epimorphism from the braid group \mathcal{B}_n into the symmetric group Sym_n . Then the kernel of π , denoted by \mathcal{PB}_n and called the *pure braid group on n strands*, is biorderable (see [24]).

The biorderable groups have the following property.

Lemma 3.1. *Let Γ be a biorderable group, and let $f, g \in \Gamma$. If $f^m = g^m$ for some $m \geq 1$, then $f = g$.*

Proof. The fact that $<$ is invariant by left and right multiplications implies that $f^m < g^m$ if $f < g$, and $f^m > g^m$ if $f > g$. So, if $f^m = g^m$, then $f = g$. \square

Now, the aim of the present subsection is to prove the following.

Proposition 3.2. *Let Σ be a surface of genus 0. Then $\mathcal{M}(\Sigma)$ is biorderable. In particular, if $f, g \in \mathcal{M}(\Sigma)$ are such that $f^m = g^m$ for some $m \geq 1$, then $f = g$.*

Remark. A group Γ is *left-orderable* if there exists a linear ordering $<$ on Γ invariant by left multiplication. By [30], the mapping class group $\mathcal{M}(\Sigma, \mathcal{P})$ is left-orderable if $\partial\Sigma \neq \emptyset$. If $\partial\Sigma = \emptyset$, then $\mathcal{M}(\Sigma, \mathcal{P})$ is not left-orderable since it has torsion, and a left-orderable group cannot have torsion.

The following lemma is a preliminary to the proof of Proposition 3.2.

Lemma 3.3. *Let $1 \rightarrow A \rightarrow B \xrightarrow{\varphi} C \rightarrow 1$ be an exact sequence of groups such that $A \simeq \mathbb{Z}^q$ for some $q \geq 0$, A is contained in the center of B , and C is biorderable. Then B is biorderable, too.*

Proof. Let $<_C$ be a linear ordering on C invariant by left and right multiplications, and let $<_A$ be a linear ordering on A invariant by (left and right) multiplications (take the lexicographic ordering, for example). Define the linear ordering $<_B$ on B by:

$$\beta_1 <_B \beta_2 \text{ if either } \varphi(\beta_1) <_C \varphi(\beta_2), \text{ or } \varphi(\beta_1) = \varphi(\beta_2) \text{ and } \beta_1 \beta_2^{-1} <_A 1.$$

It is easily checked that $<_B$ is invariant by left and right multiplications. \square

Proof of Proposition 3.2. Let Σ be a surface of genus 0 and q boundary components, c_1, \dots, c_q . If $q \leq 1$, then $\mathcal{M}(\Sigma) = \{1\}$, thus we may assume $q \geq 2$. Let $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the standard disk. For all $1 \leq i \leq q-1$, we take a copy \mathbb{D}_i of \mathbb{D} and we denote by $d_i : \mathbb{S}^1 \rightarrow \mathbb{D}_i$ its boundary component. Define the surface $\Sigma_0 = (\Sigma \sqcup (\sqcup_{1 \leq i \leq q-1} \mathbb{D}_i)) / \sim$, where \sim is the equivalence relation which identifies $c_i(z)$ with $d_i(\bar{z}) = d_i^{-1}(z)$ for all $1 \leq i \leq q-1$ and all $z \in \mathbb{S}^1$. Note that Σ_0 is (topologically) a disk and its unique boundary component is c_q . We choose a point Q_i in the interior of \mathbb{D}_i for all $1 \leq i \leq q-1$, and we set $\mathcal{Q} = \{Q_1, \dots, Q_{q-1}\}$. The group $\mathcal{M}(\Sigma_0, \mathcal{Q})$ is (isomorphic to) the braid group \mathcal{B}_{q-1} on $q-1$ strands. Moreover, the embedding $\Sigma \rightarrow \Sigma_0$ determines a homomorphism $\theta : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{Q}) = \mathcal{B}_{q-1}$ whose image is the pure braid group \mathcal{PB}_{q-1} , a biorderable group (see [24]). Let K be the subgroup of $\mathcal{M}(\Sigma)$ generated by $\{\tau_{c_1}, \dots, \tau_{c_{q-1}}\}$. Then, by Proposition 2.1, the kernel of θ is K , it is a free abelian group of rank $q-1$, and it is contained in the center of $\mathcal{M}(\Sigma)$ (see [29]). We conclude by Lemma 3.3 that $\mathcal{M}(\Sigma)$ is biorderable. \square

3.2 Genus 1 surfaces

Let \mathbb{T}^2 denote the 2-dimensional torus. Let a, b be the closed curves pictured in Figure 3.1, and let $\alpha = \tau_a \tau_b$, $\beta = \tau_a \tau_b \tau_a$, and $\delta = (\tau_a \tau_b)^3$. Then $\mathcal{M}(\mathbb{T}^2)$ is well-known to be isomorphic to $SL_2(\mathbb{Z})$ and has a presentation with generators α, β, δ and relations

$$\alpha^3 = \beta^2 = \delta, \quad \delta^2 = 1.$$

The center $Z(\mathcal{M}(\mathbb{T}^2))$ of $\mathcal{M}(\mathbb{T}^2)$ is the cyclic group $\{1, \delta\}$ of order 2, and $\mathcal{M}(\mathbb{T}^2)/Z(\mathcal{M}(\mathbb{T}^2)) = \mathcal{C}_3 * \mathcal{C}_2$ is the free product of $\mathcal{C}_3 = \{1, \bar{\alpha}, \bar{\alpha}^2\}$, a cyclic group of order 3, and $\mathcal{C}_2 = \{1, \bar{\beta}\}$, a cyclic group of order 2.

Using the fact that any finite order element of $\mathcal{C}_3 * \mathcal{C}_2$ is conjugate to an element of either \mathcal{C}_3 or \mathcal{C}_2 (see [31]), one can easily prove the following.

Lemma 3.4. *There is a unique element of order 2 in $\mathcal{M}(\mathbb{T}^2)$: δ . There are two conjugacy classes of elements of order 4 represented by β and β^{-1} . There are two conjugacy classes of*

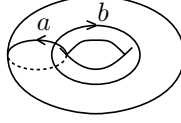


Figure 3.1. The torus \mathbb{T}^2 .

elements of order 3 represented by α^2 and α^{-2} . There are two conjugacy classes of elements of order 6 represented by α and α^{-1} . These are all the conjugacy classes of elements of finite order in $\mathcal{M}(\mathbb{T}^2)$. \square

A m -root of a given element of infinite order in $\mathcal{M}(\mathbb{T}^2)$ is not necessarily unique up to conjugation. However, such a m -root is unique up to the center as we turn now to show.

Proposition 3.5. *Let $f, g \in \mathcal{M}(\mathbb{T}^2)$ be two elements of infinite order. If $f^m = g^m$ for some $m \geq 1$, then either $f = g$ or $f = \delta g$.*

Proof. First, observe that an element $f \in SL_2(\mathbb{Z}) \simeq \mathcal{M}(\mathbb{T}^2)$ has infinite order if and only if it has real eigenvalues and $f \notin \{\text{Id}, -\text{Id}\}$. (The only non-trivial part of the above statement is that an element $f \in SL_2(\mathbb{Z})$ with non-real eigenvalues has finite order. In order to prove that, notice that the norm $\|f^m\|$ of any power of such a f is bounded by a constant K , and there are finitely many elements $g \in SL_2(\mathbb{Z})$ such that $\|g\| \leq K$.)

Now, take two elements $f, g \in SL_2(\mathbb{Z})$ of infinite order such that $f^m = g^m$ for some $m \geq 1$. In particular, f and g have real eigenvalues. The eigenspaces of f (resp. g) coincide with the eigenspaces of $f^m = g^m$, hence f and g have the same eigenspaces.

Case 1 : f has two distinct eigenvalues λ and $1/\lambda$. Then g has also two distinct eigenvalues μ and $1/\mu$. The equality $f^m = g^m$ implies that $\lambda^m = \mu^m$, thus $\lambda = \pm\mu$ and $f = \pm g$.

Case 2 : f has a unique eigenvalue (which is equal to 1 or -1). Then there is a basis of \mathbb{R}^2 (indeed of \mathbb{Z}^2) relatively to which f and g have the forms $\begin{pmatrix} \varepsilon_f & \nu_f \\ 0 & \varepsilon_f \end{pmatrix}$ and $\begin{pmatrix} \varepsilon_g & \nu_g \\ 0 & \varepsilon_g \end{pmatrix}$, respectively, where $\varepsilon_f, \varepsilon_g \in \{\pm 1\}$. Now, it is easily deduced from the equality $f^m = g^m$ that $f = \pm g$. \square

We turn now to study the mapping class groups of the genus 1 surfaces with non-empty boundary. Our main result is the following.

Theorem 3.6. *Let Σ be a surface of genus 1 with non-empty boundary. Let $f, g \in \mathcal{M}(\Sigma)$. If $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate.*

The proof of Theorem 3.6 is long and tedious, so we put it separately in the last section and go directly to the next section and the study of the mapping class groups of the high genus surfaces.

4 Surfaces with non-empty boundary

We start with a simple lemma which will be used later.

Lemma 4.1. *Let Σ be a closed surface (i.e. $\partial\Sigma = \emptyset$), and let \mathcal{P} be a non-empty finite set of punctures in Σ . Let $P_0 \in \mathcal{P}$, and let Γ be a finite subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ such that $h(P_0) = P_0$ for all $h \in \Gamma$. Then Γ is cyclic.*

Proof. By Theorem 2.25, there exists a finite subgroup $\tilde{\Gamma} \subset \text{Diff}(\Sigma, \mathcal{P})$ which is sent isomorphically onto Γ under the natural homomorphism $\text{Diff}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$. Moreover, we have $H(P_0) = P_0$ for all $H \in \tilde{\Gamma}$. Since $\tilde{\Gamma}$ is finite, there exists a Riemannian metric g on Σ which is invariant under the action of $\tilde{\Gamma}$, that is, $\tilde{\Gamma} \subset \text{Isom}_+(\Sigma, g)$. Now, the fact that $H(P_0) = P_0$ for all $H \in \tilde{\Gamma}$ implies that $\tilde{\Gamma}$ is a finite subgroup of $O_+(T_{P_0}\Sigma) \simeq \mathbb{S}^1$, hence $\tilde{\Gamma}$ is cyclic. \square

Now, the bad news of the section are the following.

Proposition 4.2. *Let Σ be a surface of genus $\rho \geq 2$ and 1 boundary component, c . Then there exist two periodic elements $f, g \in \mathcal{M}(\Sigma)$ such that $f^2 = g^2 = \tau_c$, f and g are not conjugate, and none of the conjugates of f commutes with g .*

Proof. Let $\mathbb{A} = [0, 1] \times \mathbb{S}^1$ denote the standard annulus. Let $a_k, a'_k : [0, 1] \rightarrow \partial\mathbb{A}$, $1 \leq k \leq 2\rho$, be the arcs defined by

$$a_k(t) = (1, \exp(\frac{i\pi}{2\rho}(k-1+t))), \quad a'_k(t) = (1, -\exp(\frac{i\pi}{2\rho}(k-t))).$$

(See Figure 4.1.) Then Σ can be presented as the quotient $\Sigma = \mathbb{A} / \sim_f$, where \sim_f is the equivalence relation which identifies $a_k(t)$ with $a'_k(t)$ for all $1 \leq k \leq 2\rho$ and all $t \in [0, 1]$. Let $\pi_f : \mathbb{A} \rightarrow \Sigma$ be the natural quotient, and, for $1 \leq k \leq 2\rho$, let $\bar{a}_k = \pi_f \circ a_k = \pi_f \circ a'_k$. Then the \bar{a}_k 's are loops in Σ and $\{\bar{a}_k; 1 \leq k \leq 2\rho\}$ is a basis for $H_1(\Sigma, \mathbb{R}) \simeq \mathbb{R}^{2\rho}$, where $[\bar{a}_k]$ denotes the homology class of \bar{a}_k . We choose a smooth map $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(0) = \varphi'(1) = 0$, and $\varphi'(t) \geq 0$ for all $t \in [0, 1]$, and we define $F_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}$ by

$$F_{\mathbb{A}}(t, z) = (t, \exp(i\pi\varphi(t)) \cdot z).$$

Then $F_{\mathbb{A}}$ induces a diffeomorphism $F \in \text{Diff}(\Sigma)$ via π_f , and we define $f \in \mathcal{M}(\Sigma)$ as the isotopy class of F . Note that $F \circ \bar{a}_k = \bar{a}_k^{-1}$ for all $1 \leq k \leq 2\rho$, thus the spectrum of the action of f on $H_1(\Sigma, \mathbb{R})$ is $(-1, \dots, -1)$. Moreover, we have $f^2 = \tau_c$.

Let $b_j, c_j, b'_j, c'_j : [0, 1] \rightarrow \partial\mathbb{A}$, $j = 1, 2$, be the arcs defined by

$$\begin{aligned} b_j(t) &= (1, \exp(\frac{i\pi}{2\rho}(2j\rho - 4 + t))), & c_j(t) &= (1, \exp(\frac{i\pi}{2\rho}(2j\rho - 3 + t))), \\ b'_j(t) &= (1, \exp(\frac{i\pi}{2\rho}(2j\rho - 1 - t))), & c'_j(t) &= (1, \exp(\frac{i\pi}{2\rho}(2j\rho - t))). \end{aligned}$$

(See Figure 4.2.) The surface Σ can also be presented as the quotient $\Sigma = \mathbb{A} / \sim_g$, where \sim_g is the equivalence relation which identifies $a_k(t)$ with $a'_k(t)$ for all $1 \leq k \leq 2\rho - 4$ and all $t \in [0, 1]$, and which identifies $b_j(t)$ with $b'_j(t)$, and $c_j(t)$ with $c'_j(t)$, for all $j \in \{1, 2\}$ and all $t \in [0, 1]$.

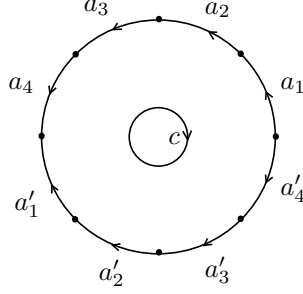


Figure 4.1. A surface of genus 2 with 1 boundary component.

Let $\pi_g : \mathbb{A} \rightarrow \Sigma$ be the natural quotient, let $\bar{a}_k = \pi_g \circ a_k = \pi_g \circ a'_k$ for $1 \leq k \leq 2\rho - 4$, and let $\bar{b}_j = \pi_g \circ b_j = \pi_g \circ b'_j$ and $\bar{c}_j = \pi_g \circ c_j = \pi_g \circ c'_j$ for $j = 1, 2$. Then all the \bar{a}_k 's, the \bar{b}_j 's, and the \bar{c}_j 's are loops in Σ , and the set $\{\bar{a}_k; 1 \leq k \leq 2\rho - 4\} \cup \{\bar{b}_1, \bar{b}_2, \bar{c}_1, \bar{c}_2\}$ is a basis for $H_1(\Sigma, \mathbb{R})$. Here again, $F_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}$ induces a diffeomorphism $G \in \text{Diff}(\Sigma)$ via π_g , and we define $g \in \mathcal{M}(\Sigma)$ as the isotopy class of G . Note that $G \circ \bar{a}_k = \bar{a}_k^{-1}$ for all $1 \leq k \leq 2\rho - 4$, $G \circ \bar{b}_1 = \bar{b}_2$, $G \circ \bar{b}_2 = \bar{b}_1$, $G \circ \bar{c}_1 = \bar{c}_2$, $G \circ \bar{c}_2 = \bar{c}_1$, thus the spectrum of the action of g on $H_1(\Sigma, \mathbb{R})$ is $(-1, \dots, -1, -1, -1, 1, 1)$. Moreover, we have $g^2 = \tau_c$.

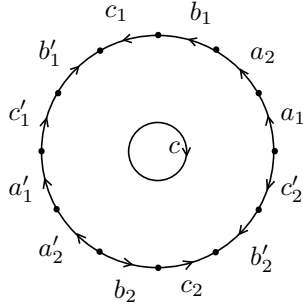


Figure 4.2. A surface of genus 3 with 1 boundary component.

We already know that $f^2 = g^2 = \tau_c$. On the other hand, the spectrum of the action of f on $H_1(\Sigma, \mathbb{R})$ is different from the spectrum of the action of g , thus f and g are not conjugate. Suppose that there exists $f' \in \mathcal{M}(\Sigma)$ conjugate to f and commuting with g . Let $\theta : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{Q})$ be the corking of Σ . Then $\theta(f')(Q_1) = \theta(g)(Q_1) = Q_1$, and $\{\theta(f'), \theta(g)\}$ generates a subgroup of $\mathcal{M}(\Sigma_0, \mathcal{Q})$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This is a finite subgroup of $\mathcal{M}(\Sigma_0, \mathcal{Q})$ whose elements fix Q_1 and which is not cyclic, contradicting Lemma 4.1. \square

Corollary 4.3. *Let Σ be a surface of genus $\rho \geq 2$ and q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite set of punctures in the interior of Σ . Then there exist $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $f^2 = g^2$, f and g are not conjugate, and none of the conjugates of f commutes with g .*

Proof. Obviously, we can assume that either $q \geq 2$ or $\mathcal{P} \neq \emptyset$. (The case $q = 1$ and $\mathcal{P} = \emptyset$ is covered by the conclusion of Proposition 4.2.) Let Σ_1 be a surface of genus ρ and 1 boundary

component, $c^{(1)}$, and let Σ_2 be a surface of genus 0 and $q+1$ boundary components, $c_1, \dots, c_q, c^{(2)}$. The surface Σ can be presented as the quotient $\Sigma = (\Sigma_1 \sqcup \Sigma_2) / \sim$, where \sim is the equivalence relation which identifies $c^{(1)}(z)$ with $c^{(2)}(\bar{z}) = (c^{(2)})^{-1}(z)$ for all $z \in \mathbb{S}^1$ (see Figure 4.3). We denote by $\pi : \Sigma_1 \sqcup \Sigma_2 \rightarrow \Sigma$ the natural quotient, we assume that \mathcal{P} is contained in the interior of $\pi(\Sigma_2)$, and we set $c = \pi \circ c^{(1)} = \pi \circ (c^{(2)})^{-1}$ and $\gamma = \langle c \rangle$. Then π determines a homomorphism $\theta : \mathcal{M}(\Sigma_1) \times \mathcal{M}(\Sigma_2, \pi^{-1}(\mathcal{P})) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$ whose image is $\text{Stab}(\gamma) = \{h \in \mathcal{M}(\Sigma, \mathcal{P}); h(\gamma) = \gamma\}$. Furthermore, the restriction of θ to $\mathcal{M}(\Sigma_1)$ is injective (see [29]).

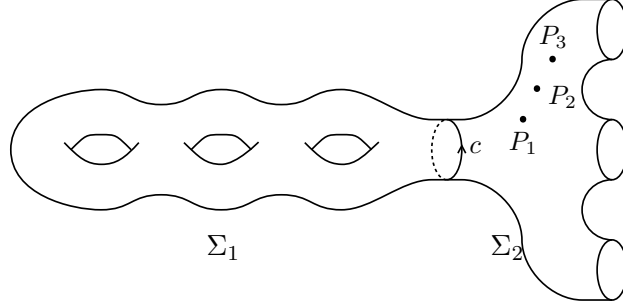


Figure 4.3. Cutting Σ into two pieces.

By Proposition 4.2, there exist two periodic elements $f_1, g_1 \in \mathcal{M}(\Sigma_1)$ such that $f_1^2 = g_1^2 = \tau_{c^{(1)}}$, f_1 and g_1 are not conjugate in $\mathcal{M}(\Sigma_1)$, and none of the conjugates of f_1 commutes with g_1 . We set $f = \theta(f_1)$ and $g = \theta(g_1)$. Then $f^2 = g^2 = \tau_c$ and $\Delta(f) = \Delta(g) = \{\gamma\}$.

Suppose that there exists $h \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $hfh^{-1} = g$. We have

$$\{\gamma\} = \Delta(g) = \Delta(hfh^{-1}) = h(\Delta(f)) = h(\{\gamma\}),$$

thus $h \in \text{Stab}(\gamma)$. Let $(h_1, h_2) \in \mathcal{M}(\Sigma_1) \times \mathcal{M}(\Sigma_2, \pi^{-1}(\mathcal{P}))$ such that $h = \theta(h_1, h_2)$. Then

$$\theta(h_1 f_1 h_1^{-1}) = \theta((h_1, h_2) \cdot (f_1, \text{Id}) \cdot (h_1^{-1}, h_2^{-1})) = hfh^{-1} = g = \theta((g_1, \text{Id})) = \theta(g_1),$$

thus $h_1 f_1 h_1^{-1} = g_1$ (since $\theta|_{\mathcal{M}(\Sigma_1)}$ is injective): a contradiction.

Suppose that there exists $h \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $(hfh^{-1})g = g(hfh^{-1})$. Set $\gamma' = h(\gamma)$ and choose an essential curve $c' : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ which represents γ' . We have $g^2 = \tau_c$, $hf^2h^{-1} = h\tau_ch^{-1} = \tau_{c'}$, and g commutes with hfh^{-1} , thus τ_c and $\tau_{c'}$ commute, therefore c' can be chosen so that $c' \cap c = \emptyset$ (see [29], for example). Both, c and c' , separate the surface Σ into two subsurfaces, one of genus ρ with one boundary component, and one of genus 0 with $q+1$ boundary components and containing the whole set \mathcal{P} . Obviously, since $c' \cap c = \emptyset$, this is possible only if c' and c are isotopic, that is, $\gamma' = \gamma$. So, $h \in \text{Stab}(\gamma)$. Let $(h_1, h_2) \in \mathcal{M}(\Sigma_1) \times \mathcal{M}(\Sigma_2, \pi^{-1}(\mathcal{P}))$ such that $h = \theta(h_1, h_2)$. Then $(h_1 f_1 h_1^{-1})g_1 = g_1(h_1 f_1 h_1^{-1})$: a contradiction. \square

Remark. If q is large enough, then the elements f and g in the above corollary cannot be chosen to be periodic because of the following.

Lemma 4.4. *Let Σ be a surface of genus $\rho \geq 2$ and q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite set of punctures in the interior of Σ . Let K be the*

subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ generated by $\{\tau_{c_1}, \dots, \tau_{c_q}\}$. If $q > 2\rho + 2$, then all the periodic elements of $\mathcal{M}(\Sigma, \mathcal{P})$ belong to K . In particular, if $q > 2\rho + 2$, and if $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ are two periodic elements such that $f^m = g^m$ for some $m \geq 1$, then $f = g$.

Proof. Let $\theta : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ be the corking of (Σ, \mathcal{P}) . Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$ be a periodic element such that $\theta(f) \notin K$. This means that $\theta(f)$ is of finite order $m \geq 2$. Let $F \in \text{Diff}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ be a diffeomorphism of order m which represents $\theta(f)$. Since $F(Q_i) = Q_i$ for all $Q_i \in \mathcal{Q}$, by Lemma 2.27, we have

$$\begin{aligned} & \chi(\Sigma_0) + q(m-1) \leq m \cdot \chi(\Sigma_0/F) \\ \Rightarrow & 2 - 2\rho + q(m-1) \leq 2m & (\text{since } \chi(\Sigma_0/F) \leq 2) \\ \Rightarrow & 2 - 2\rho - q + (q-2)m \leq 0 \\ \Rightarrow & 2 - 2\rho - q + 2(q-2) \leq 0 & (\text{since } m \geq 2) \\ \Rightarrow & q \leq 2\rho + 2. \end{aligned}$$

The uniqueness of a m -root of a given periodic element in $\mathcal{M}(\Sigma, \mathcal{P})$, if $q > 2\rho + 2$, is simply due to the fact that $K \simeq \mathbb{Z}^q$ and such a root must lie in K . \square

We turn now to the good news of the section.

Theorem 4.5. *Let Σ be a surface of genus $\rho \geq 0$ and q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite set of punctures in the interior of Σ . Let $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ be two pseudo-Anosov elements. If $f^m = g^m$ for some $m \geq 1$, then $f = g$.*

Proof. Let $\theta : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ be the corking of (Σ, \mathcal{P}) . By Corollary 2.14, there are pseudo-Anosov representatives $F, G \in \text{Diff}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ of $\theta(f), \theta(g)$, respectively, such that $F^m = G^m$. Let \mathcal{F}^s (resp. \mathcal{F}^u) be the stable (resp. unstable) foliation of F , and let λ be its dilatation coefficient. Then \mathcal{F}^s (resp. \mathcal{F}^u) is also the stable (resp. unstable) foliation of G and λ is its dilatation coefficient. This implies that $F \circ G \circ F^{-1} \circ G^{-1} \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$.

Fix some $Q_1 \in \mathcal{Q}$. Let \mathcal{SS}_{Q_1} be the set of separatrices of \mathcal{F}^s at Q_1 . We number $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_k$ the elements of \mathcal{SS}_{Q_1} so that the corresponding prongs are anticlockwise numbered. Let $\text{Norm}_{Q_1}(\mathcal{F}^s, \mathcal{F}^u)$ be the subgroup of $\text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ of elements $H \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ that fix Q_1 . All the elements of $\text{Norm}_{Q_1}(\mathcal{F}^s, \mathcal{F}^u)$ permute the set \mathcal{SS}_{Q_1} , and this determines a well-defined homomorphism $\psi : \text{Norm}_{Q_1}(\mathcal{F}^s, \mathcal{F}^u) \rightarrow \text{Sym}(\mathcal{SS}_{Q_1})$. Moreover, if $\omega \in \text{Sym}(\mathcal{SS}_{Q_1})$ denotes the cyclic permutation of $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_k$, then $\psi(H)$ is a power of ω for all $H \in \text{Norm}_{Q_1}(\mathcal{F}^s, \mathcal{F}^u)$. In particular, if $H_1, H_2 \in \text{Norm}_{Q_1}(\mathcal{F}^s, \mathcal{F}^u)$, then $\psi(H_1 \circ H_2 \circ H_1^{-1} \circ H_2^{-1}) = 1$.

Note that $F, G \in \text{Norm}_{Q_1}(\mathcal{F}^s, \mathcal{F}^u)$, thus, by the above observation, we have $\psi(F \circ G \circ F^{-1} \circ G^{-1}) = 1$. Furthermore, we already know that $F \circ G \circ F^{-1} \circ G^{-1} \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$, thus, by Lemma 2.9, $F \circ G \circ F^{-1} \circ G^{-1} = \text{Id}$. So, $\theta(fgf^{-1}g^{-1}) = 1$.

Let K be the subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ generated by $\tau_{c_1}, \dots, \tau_{c_q}$. By Proposition 2.1, there exists $u \in K$ such that $fgf^{-1} = ug$. Since u is central, it follows that

$$g = g^m g g^{-m} = f^m g f^{-m} = u^m g,$$

thus $u^m = 1$. Since K is torsion free, we deduce that $u = 1$, that is, f and g commute.

Set $h = gf^{-1}$. Then

$$g^m = (hf)^m = h^m f^m = h^m g^m,$$

thus $h^m = 1$, therefore $h = 1$ (since $\mathcal{M}(\Sigma, \mathcal{P})$ is torsion free), that is, $f = g$. \square

5 Closed surfaces

The good news of this section are the following.

Proposition 5.1. *Let Σ be a closed surface of genus $\rho \geq 2$, and let \mathcal{P} be a finite set of punctures in Σ . Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$ be a pseudo-Anosov element. Then the set of $g \in \mathcal{M}(\Sigma, \mathcal{P})$ satisfying $g^m = f^m$ for some $m \geq 1$ is finite.*

Proof. We take a pseudo-Anosov representative $F \in \text{Diff}(\Sigma, \mathcal{P})$ of f , we denote by \mathcal{F}^s (resp. \mathcal{F}^u) the stable (resp. unstable) foliation of F , we denote by λ its dilatation coefficient, and we set $\tilde{\Gamma}_0 = \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$. The subgroup $\tilde{\Gamma}_0$ is finite by Corollary 2.10 and, by Corollary 2.12, it is sent isomorphically onto a subgroup Γ_0 of $\mathcal{M}(\Sigma, \mathcal{P})$ under the natural homomorphism $\text{Diff}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$.

Set $\mathcal{U} = \{g \in \mathcal{M}(\Sigma, \mathcal{P}); g^m = f^m \text{ for some } m \geq 1\}$. Let $g \in \mathcal{U}$, and let $m \geq 1$ such that $g^m = f^m$. Clearly, this last equality implies that g is a pseudo-Anosov element of $\mathcal{M}(\Sigma, \mathcal{P})$. By Corollary 2.14, there exists a pseudo-Anosov representative $G \in \text{Diff}(\Sigma, \mathcal{P})$ of g such that $G^m = F^m$. Clearly, \mathcal{F}^s (resp. \mathcal{F}^u) is the stable (resp. unstable) foliation of G , and λ is its dilatation coefficient, hence $F^{-1}G \in \tilde{\Gamma}_0 = \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$, therefore $f^{-1}g \in \Gamma_0$. This shows that $\mathcal{U} \subset f \cdot \Gamma_0 = \{fh; h \in \Gamma_0\}$, hence \mathcal{U} is finite. \square

However, the situation can be pretty bad as we turn now to show.

Theorem 5.2.

1. *Let Σ be a closed surface of genus $\rho \geq 2$. Then there exist two pseudo-Anosov elements $f, g \in \mathcal{M}(\Sigma)$ such that $f^{2(\rho+1)} = g^{2(\rho+1)}$, f is not conjugate to g , and none of the conjugates of f commutes with g .*
2. *Let Σ be a closed surface of genus $\rho \geq 4$, with $\rho \equiv 0 \pmod{4}$. Then there exist two pseudo-Anosov elements $f, g \in \mathcal{M}(\Sigma)$ such that $f^2 = g^2$, f is not conjugate to g , and none of the conjugates of f commutes with g .*

Proof of Theorem 5.2.1. We consider a closed surface Σ of genus $\rho \geq 2$. This admits a cellular decomposition (see Figure 5.1) with $2(\rho + 1)$ vertices, $P_1, P_2, \dots, P_{2(\rho+1)}$, with $4(\rho + 1)$ arrows, $a_1, a_2, \dots, a_{\rho+1}, b_1, b_2, \dots, b_{\rho+1}, c_1, c_2, \dots, c_{\rho+1}, d_1, d_2, \dots, d_{\rho+1}$, and with 4 faces, A_1, A_2, A_3, A_4 , where the sources and the targets of the arrows are

$$\begin{aligned} \text{source}(a_i) &= \text{source}(d_i) = P_{2i-1}, & \text{target}(a_i) &= \text{target}(d_i) = P_{2i}, \\ \text{source}(b_i) &= \text{source}(c_i) = P_{2i}, & \text{target}(b_i) &= \text{target}(c_i) = P_{2i+1}, \end{aligned}$$

for all $1 \leq i \leq \rho + 1$, and where the boundaries of the faces are

$$\begin{aligned} \partial A_1 &= a_1 b_1 a_2 b_2 \dots a_{\rho+1} b_{\rho+1}, & \partial A_2 &= c_{\rho+1}^{-1} a_{\rho+1}^{-1} \dots c_2^{-1} a_2^{-1} c_1^{-1} a_1^{-1}, \\ \partial A_3 &= b_{\rho+1}^{-1} d_{\rho+1}^{-1} \dots b_2^{-1} d_2^{-1} b_1^{-1} d_1^{-1}, & \partial A_4 &= d_1 c_1 d_2 c_2 \dots d_{\rho+1} c_{\rho+1}. \end{aligned}$$

Moreover, we endow Σ with the hyperbolic metric so that each A_j is isometric to a right-angled regular $2(\rho + 1)$ -gon in the hyperbolic plane.

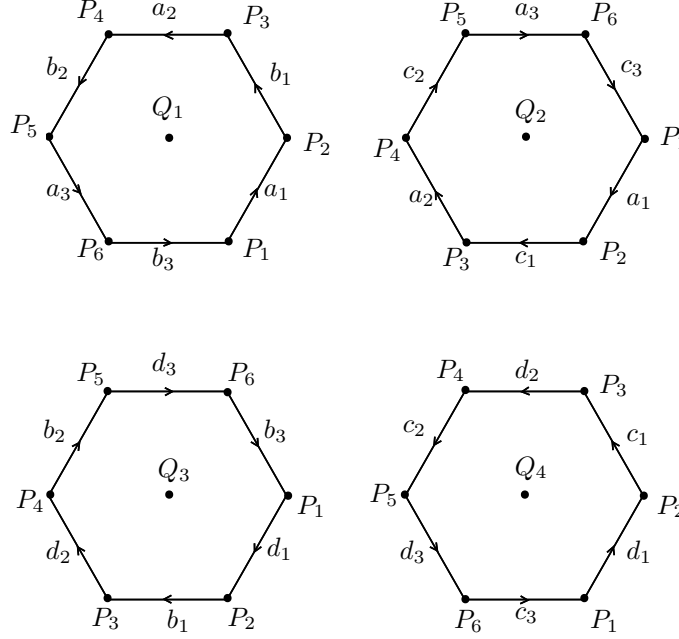


Figure 5.1. A cellular decomposition of the genus 2 oriented surface.

Consider the isometries $R, S, T \in \text{Isom}(\Sigma)$ defined as follows. The transformation R is an order $\rho + 1$ isometry such that $R(A_j) = A_j$ for all $1 \leq j \leq 4$, and whose restriction to each A_j is a rotation of angle $\frac{2\pi}{\rho+1}$ (resp. $-\frac{2\pi}{\rho+1}$) centered at Q_j for $j = 1, 4$ (resp. for $j = 2, 3$), where Q_j denotes the center of A_j . Note that R permutes cyclically $a_1, a_2, \dots, a_{\rho+1}$, as well as $b_1, b_2, \dots, b_{\rho+1}$, and $c_1, c_2, \dots, c_{\rho+1}$, and $d_1, d_2, \dots, d_{\rho+1}$. The isometry S is the (unique) involution which sends a_1 onto a_1^{-1} , d_1 onto d_1^{-1} , A_1 onto A_2 , and A_4 onto A_3 . It fixes the centers of a_1 and d_1 , and may fix two other points, depending on the evenness of $\rho + 1$. The isometry T is the (unique) involution which sends a_i onto d_i , b_i onto c_i , for all $1 \leq i \leq \rho + 1$, A_1 onto A_4 , and A_2 onto A_3 . It fixes all the P_j 's. Let Γ be the subgroup of $\text{Isom}(\Sigma)$ generated by $\{R, S, T\}$. It is easily checked that Γ has the presentation

$$\Gamma = \langle R, S, T \mid R^{\rho+1} = S^2 = T^2 = 1, S \circ R \circ S = R^{-1}, T \circ R = R \circ T, T \circ S = S \circ T \rangle,$$

and is isomorphic to $\mathcal{D}_{2(\rho+1)} \times \mathcal{C}_2$, where $\mathcal{D}_{2(\rho+1)}$ denotes the dihedral group of order $2(\rho + 1)$ and $\mathcal{C}_2 = \{\pm 1\}$.

Let $\Sigma_0 = \Sigma/\Gamma$ and let $\pi : \Sigma \rightarrow \Sigma_0$ be the natural quotient. Then Σ_0 is a sphere and π is a ramified covering with 4 ramification points, $\bar{P}_0, \bar{P}_1, \bar{P}_2, \bar{P}_3$. Up to permutation, we can assume that $\pi^{-1}(\bar{P}_0) = \{Q_1, Q_2, Q_3, Q_4\}$, $\pi^{-1}(\bar{P}_1) = \{P_1, P_2, \dots, P_{\rho+1}\}$, $\pi^{-1}(\bar{P}_2)$ is the set of the centers of the a_i 's and of the d_i 's, and $\pi^{-1}(\bar{P}_3)$ is the set of the centers of the b_i 's and of the c_i 's. In particular, we have $r_\pi(\bar{P}_0) = \rho + 1$ and $r_\pi(\bar{P}_1) = r_\pi(\bar{P}_2) = r_\pi(\bar{P}_3) = 2$.

Set $\mathcal{P}_0 = \{\bar{P}_0, \bar{P}_1, \bar{P}_2, \bar{P}_3\}$ and take a pseudo-Anosov diffeomorphism $F_0 \in \text{Diff}(\Sigma_0, \mathcal{P}_0)$. Such a diffeomorphism can be easily constructed as follows. Choose a matrix $M \in SL_2(\mathbb{Z})$ having two distinct real eigenvalues λ and $\frac{1}{\lambda}$, with $|\lambda| > 1$, and denote by F_M the Anosov diffeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by M . Let $F_{-\text{Id}}$ be the involution of \mathbb{T}^2 induced by $-\text{Id} \in SL_2(\mathbb{Z})$. Then $F_{-\text{Id}}$ commutes with F_M , thus F_M induces a diffeomorphism F_0 on $\Sigma_0 = \mathbb{T}^2/F_{-\text{Id}}$. Moreover, the projection $\mathbb{T}^2 \rightarrow \mathbb{T}^2/F_{-\text{Id}} = \Sigma_0$ is a 2-folds ramified covering of Σ_0 with 4 ramification points that we can assume to be $\bar{P}_0, \bar{P}_1, \bar{P}_2, \bar{P}_3$, thus F_0 is a pseudo-Anosov diffeomorphism of $(\Sigma_0, \mathcal{P}_0)$.

Let \mathcal{F}_0^s (resp. \mathcal{F}_0^u) denote the stable (resp. unstable) foliation of F_0 , and let \mathcal{F}^s (resp. \mathcal{F}^u) denote the lift of \mathcal{F}_0^s (resp. \mathcal{F}_0^u) on Σ . Upon replacing F_0 by some power, we can assume that F_0 lifts to a pseudo-Anosov diffeomorphism $F : \Sigma \rightarrow \Sigma$ whose stable (resp. unstable) foliation is \mathcal{F}^s (resp. \mathcal{F}^u), and which fixes its separatrices (see Subsections 2.5 and 2.6).

It is easily checked from Proposition 2.2 that $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_i) = 1$ for all $0 \leq i \leq 4$, and that the singularities of $(\mathcal{F}_0^s, \mathcal{F}_0^u)$ are precisely $\bar{P}_0, \bar{P}_1, \bar{P}_2, \bar{P}_3$. On the other hand, we already know that $r_\pi(\bar{P}_0) = \rho + 1$ and $r_\pi(\bar{P}_i) = 2$ for all $1 \leq i \leq 3$. So, \bar{P}_0 is a pivot. By Proposition 2.18, it follows that $\Gamma = \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$.

Set $F_1 = F \circ R$ and $F_2 = F \circ S$, and denote by $f_1 \in \mathcal{M}(\Sigma)$ (resp. $f_2 \in \mathcal{M}(\Sigma)$) the isotopy class of F_1 (resp. F_2). By Lemma 2.19, R and S commute with F , thus $F_1^{2(\rho+1)} = F^{2(\rho+1)} \circ R^{2(\rho+1)} = F^{2(\rho+1)} = F^{2(\rho+1)} \circ S^{2(\rho+1)} = F_2^{2(\rho+1)}$, therefore $f_1^{2(\rho+1)} = f_2^{2(\rho+1)}$.

Suppose that f_1 is conjugate to f_2 . By Proposition 2.20, there exists $H \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ R \circ H^{-1} = S$. But this equality is impossible because R is of order $\rho + 1$ while S is of order 2.

Suppose that there exists a conjugate of f_1 which commutes with f_2 . By Proposition 2.21, there exists $H \in \tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ R \circ H^{-1}$ commutes with S . Recall that Γ is isomorphic to $\mathcal{D}_{2(\rho+1)} \times \mathcal{C}_2$. Moreover, $H \circ R \circ H^{-1}$ is an element of $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u) \simeq \Gamma$ (by Lemma 2.7), and it is of order $\rho + 1$ (because R is of order $\rho + 1$). A direct inspection on the group $\Gamma \simeq \mathcal{D}_{2(\rho+1)} \times \mathcal{C}_2$ shows that the only elements of Γ of order $\rho + 1$ are:

- R^k , with k and $\rho + 1$ coprime,
- $R^k \circ T$, with $\rho + 1$ even, and k and $\rho + 1$ coprime,
- $R^k \circ T$, with $\rho + 1 \equiv 2 \pmod{4}$, and k and $\frac{\rho+1}{2}$ coprime.

It is easily checked that none of these elements commutes with S : a contradiction. \square

The following lemma gives a simple way for constructing pseudo-Anosov braids. It is also a preliminary to the proof of Theorem 5.2.2.

Lemma 5.3. *Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a set of n punctures in the sphere \mathbb{S}^2 , where n is a prime number, and let $f \in \mathcal{M}(\mathbb{S}^2, \mathcal{P})$ be a non-periodic element which permutes cyclically P_1, P_2, \dots, P_n . Then f is a pseudo-Anosov element.*

Proof. Let $f \in \mathcal{M}(\mathbb{S}^2, \mathcal{P})$ which is neither periodic, nor pseudo-Anosov (so, is reducible), and assume that it acts cyclically on P_1, P_2, \dots, P_n . The hypothesis that f is reducible means that $\Delta(f) \neq \emptyset$. Set $\Delta(f) = \{\delta_0, \delta_1, \dots, \delta_p\}$. We choose essential curves d_0, d_1, \dots, d_p such that $\langle d_i \rangle = \delta_i$ for all $0 \leq i \leq p$, and $d_i \cap d_j = \emptyset$ for all $0 \leq i \neq j \leq p$. We denote by Σ_Δ the surface obtained from \mathbb{S}^2 cutting along the d_i 's, and by $\pi_\Delta : \Sigma_\Delta \rightarrow \Sigma$ the natural quotient. Let $\Sigma_{\Delta_1}, \dots, \Sigma_{\Delta_l}$ be the connected components of Σ_Δ , $\mathcal{P}_\Delta = \pi_\Delta^{-1}(\mathcal{P})$, $\mathcal{P}_{\Delta_i} = \mathcal{P}_\Delta \cap \Sigma_{\Delta_i}$, and $\mathcal{P}_i = \pi_\Delta(\mathcal{P}_{\Delta_i})$ for $1 \leq i \leq l$. For all $1 \leq i \leq l$ there exists $1 \leq j \leq l$ such that $f(\Sigma_{\Delta_i})$ is Σ_{Δ_j} up to isotopy, and, therefore, $f(\mathcal{P}_i) = \mathcal{P}_j$. Suppose $\mathcal{P}_i \neq \emptyset$ for $1 \leq i \leq k$ and $\mathcal{P}_i = \emptyset$ for $k+1 \leq i \leq l$. The fact that \mathbb{S}^2 is a sphere implies that $k \geq 2$ and that $|\mathcal{P}_i| \geq 2$ for some $1 \leq i \leq k$ (say $|\mathcal{P}_1| \geq 2$), and the fact that f acts transitively on \mathcal{P} implies that $|\mathcal{P}_i| = |\mathcal{P}_j|$ for all $1 \leq i, j \leq k$. It follows that $n = k \cdot |\mathcal{P}_1|$, contradicting the hypothesis that n is a prime number. \square

Proof of Theorem 5.2.2. We assume that $\rho = 4l$, where $l \geq 1$, and we consider a closed oriented surface Σ of genus ρ . We represent Σ embedded in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ as follows (see Figure 5.2). Let γ_k be the segment in $\mathbb{C} = \mathbb{C} \times \{0\} \subset \mathbb{R}^3$ which joins 0 to $K \cdot \exp(\frac{2ik\pi}{\rho})$, where K is a fixed real number greater than $2/\sin(\frac{\pi}{\rho})$, and let S_k be the circle in \mathbb{C} centered at $(K+1) \cdot \exp(\frac{2ik\pi}{\rho})$ of radius 1, for all $0 \leq k \leq 2\rho - 1$. Set

$$\Upsilon = \left(\bigcup_{k=0}^{2\rho-1} \gamma_k \right) \cup \left(\bigcup_{k=0}^{2\rho-1} S_k \right).$$

Note that Υ is a connected graph embedded in $\mathbb{C} \subset \mathbb{R}^3$. We define Σ as the boundary of a regular neighborhood of Υ .

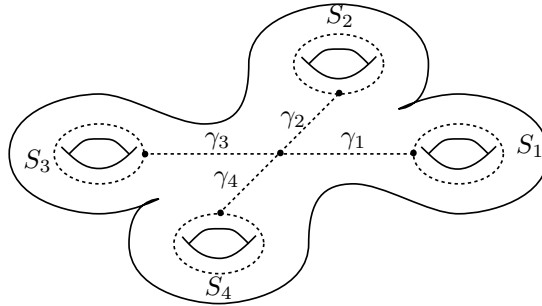


Figure 5.2. A surface of genus 4.

Consider the isometries $R, S \in \text{Isom}(\Sigma)$ defined as follows. Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis of \mathbb{R}^3 . Then R is the rotation of angle $\frac{2\pi}{\rho}$ around the axis $\mathbb{R}\vec{e}_3$, and S is the half-turn around the axis $\mathbb{R}\vec{e}_1$. Let Γ be the group generated by $\{R, S\}$. It is easily checked that Γ has the presentation

$$\Gamma = \langle R, S \mid R^\rho = S^2 = 1, S \circ R \circ S = R^{-1} \rangle,$$

and is isomorphic to the dihedral group $\mathcal{D}_{2\rho}$. Note that $R^k \circ S$ is a half-turn with 6 fixed points on Σ if k is even, $R^k \circ S$ has 2 fixed points on Σ if k is odd, and R^k has 2 fixed points on Σ if $1 \leq k \leq \rho - 1$. In particular, the elements of Γ having precisely 6 fixed points are those of the form $R^k \circ S$ with k even.

Let $\Sigma_0 = \Sigma/\Gamma$, and let $\pi : \Sigma \rightarrow \Sigma_0$ be the natural quotient. Then Σ_0 is a sphere and π is a ramified covering with 5 ramification points, $\bar{P}_0, \bar{P}_1, \bar{P}_2, \bar{P}_3, \bar{P}_4$. Up to a permutation, we can assume that $\pi^{-1}(\bar{P}_0) = \Sigma \cap \mathbb{R}\vec{e}_3$ (which has 2 points) and, consequently, that $r_\pi(\bar{P}_0) = \rho$. Note that $r_\pi(\bar{P}_i) = 2$ for the remaining indices $1 \leq i \leq 4$.

Set $\mathcal{P}_0 = \{\bar{P}_0, \bar{P}_1, \bar{P}_2, \bar{P}_3, \bar{P}_4\}$. Let $f_0 \in \mathcal{M}(\Sigma_0, \mathcal{P}_0)$ be a non-periodic element which permutes cyclically $\bar{P}_0, \bar{P}_1, \dots, \bar{P}_4$. By Lemma 5.3, f_0 is a pseudo-Anosov element. Let $F_0 \in \text{Diff}(\Sigma_0, \mathcal{P}_0)$ be a pseudo-Anosov diffeomorphism which represents f_0 , and let \mathcal{F}_0^s (resp. \mathcal{F}_0^u) be the stable (resp. unstable) foliation of F_0 . Since F_0 permutes cyclically $\bar{P}_0, \bar{P}_1, \dots, \bar{P}_4$, we have $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_i) = \text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_0)$ for all $1 \leq i \leq 4$. On the other hand, by Proposition 2.2, one of these indices must be 1, thus $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_i) = 1$ for all $0 \leq i \leq 4$. Let \mathcal{S}_0 denote the set of singularities of $(\mathcal{F}_0^s, \mathcal{F}_0^u)$. Then, again by Proposition 2.2, there exists $\bar{P}_5 \in \Sigma_0 \setminus \mathcal{P}_0$ such that $\mathcal{S}_0 = \mathcal{P}_0 \cup \{\bar{P}_5\}$ and $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_5) = 3$.

Let \mathcal{F}^s (resp. \mathcal{F}^u) be the lift of \mathcal{F}_0^s (resp. \mathcal{F}_0^u) on Σ . By Lemma 2.16, there exists some $k \geq 1$ such that F_0^k lifts to a pseudo-Anosov diffeomorphism $F : \Sigma \rightarrow \Sigma$ whose stable (resp. unstable) foliation is \mathcal{F}^s (resp. \mathcal{F}^u). Obviously, k can be chosen so that F fixes its separatrices. Recall that $r_\pi(\bar{P}_0) = \rho$ and $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_0) = 1$, $r_\pi(\bar{P}_i) = 2$ and $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_i) = 1$ for all $1 \leq i \leq 4$, and $r_\pi(\bar{P}_5) = 1$ and $\text{Ind}(\mathcal{F}_0^s, \mathcal{F}_0^u : \bar{P}_5) = 3$, thus \bar{P}_0 is a pivot. By Lemma 2.18, it follows that $\Gamma = \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$.

Set $F_1 = F \circ S$ and $F_2 = F \circ R \circ S$, and denote by $f_1 \in \mathcal{M}(\Sigma)$ (resp. $f_2 \in \mathcal{M}(\Sigma)$) the isotopy class of F_1 (resp. F_2). By Lemma 2.19, R and S commute with F , thus $F_1^2 = F^2 = F_2^2$, therefore $f_1^2 = f_2^2$.

Suppose that f_1 and f_2 are conjugate in $\mathcal{M}(\Sigma)$. By Proposition 2.20, there exists $H \in \text{Norm}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ S \circ H^{-1} = R \circ S$. But this equality is impossible because S has 6 fixed points on Σ while $R \circ S$ has only 2.

Suppose that there exists a conjugate of f_1 which commutes with f_2 . By Proposition 2.21, there exists $H \in \tilde{\text{Norm}}(\mathcal{F}^s, \mathcal{F}^u)$ such that $H \circ S \circ H^{-1}$ commutes with $R \circ S$. Observe that $H \circ S \circ H^{-1}$ should be an order 2 element of $\text{Sym}(\mathcal{F}^s, \mathcal{F}^u) = \Gamma$ which fixes 6 points on Σ , thus, as pointed out before, it is of the form $R^k \circ S$ with k even. On the other hand, the only elements of order 2 in $\Gamma \simeq \mathcal{D}_{2\rho}$ which commute with $R \circ S$ are $R \circ S$, $R^{\frac{\rho-2}{2}} \circ S$, and $R^{\frac{\rho}{2}}$, and all of them have precisely 2 fixed points on Σ (note that $\frac{\rho-2}{2}$ is odd). This is a contradiction. \square

6 Pure subgroups

Theorem 6.1. *Let Σ be a surface of genus $\rho \geq 0$ and q boundary components, c_1, \dots, c_q , and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite set of punctures in the interior of Σ . Let Γ be a pure subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$. If $f, g \in \Gamma$ are such that $f^m = g^m$ for some $m \geq 1$, then $f = g$.*

Proof. We consider two elements $f, g \in \Gamma$ such that $f^m = g^m$ for some $m \geq 1$. Obviously, we have $\Delta(f) = \Delta(g)$ and f, g are either both periodic, or both pseudo-Anosov, or both reducible.

Assume that f, g are both periodic. Let K be the subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ generated by $\{\tau_{c_1}, \dots, \tau_{c_q}\}$. The only periodic pure elements of $\mathcal{M}(\Sigma, \mathcal{P})$ are the elements of K , and $K \simeq \mathbb{Z}^q$, thus $f, g \in K$, and the equality $f^m = g^m$ clearly implies $f = g$.

Now, assume that f, g are both pseudo-Anosov. Let $\theta : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ be the corking of (Σ, \mathcal{P}) . By Corollary 2.14, there exist pseudo-Anosov representatives $F, G \in \text{Diff}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ of $\theta(f), \theta(g)$, respectively, such that $F^m = G^m$. Let \mathcal{F}^s (resp. \mathcal{F}^u) be the stable (resp. unstable) foliation of F , and let λ be its dilatation coefficient. Then \mathcal{F}^s (resp. \mathcal{F}^u) is also the stable (resp. unstable) foliation of G , and λ is its dilatation coefficient. It follows that $GF^{-1} \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$, thus GF^{-1} has finite order, therefore gf^{-1} is periodic (since GF^{-1} represents $\theta(gf^{-1})$). As pointed out before, the periodic elements of Γ are precisely the elements of $\Gamma \cap K$, thus $gf^{-1} \in K$. Set $h = gf^{-1}$. The subgroup K is contained in the center of $\mathcal{M}(\Sigma, \mathcal{P})$, thus

$$g^m = (hf)^m = h^m f^m = h^m g^m,$$

therefore $h^m = 1$. We conclude that $h = 1$ (since K is torsion free), that is, $g = f$.

Now, assume that f, g are both reducible. We use the same notations as in Subsection 2.7. So, we set $\Delta = \Delta(f) = \Delta(g) = \{\delta_0, \delta_1, \dots, \delta_p\}$, and we choose essential curves d_0, d_1, \dots, d_p of (Σ, \mathcal{P}) such that $\langle d_i \rangle = \delta_i$ for all $0 \leq i \leq p$, and $d_i \cap d_j = \emptyset$ for all $0 \leq i \neq j \leq p$. We denote by Σ_Δ the surface obtained from Σ cutting along the d_i 's. Thus, Σ_Δ is a non-necessarily connected surface whose boundary components are $c_1, \dots, c_q, d_0^{(1)}, d_1^{(1)}, \dots, d_p^{(1)}, d_0^{(2)}, d_1^{(2)}, \dots, d_p^{(2)}$ and $\Sigma = \Sigma_\Delta / \sim$, where \sim is the equivalence relation which identifies $d_i^{(1)}(z)$ with $d_i^{(2)}(\bar{z}) = (d_i^{(2)})^{-1}(z)$ for all $0 \leq i \leq p$ and all $z \in \mathbb{S}^1$. Let $\pi_\Delta : \Sigma_\Delta \rightarrow \Sigma$ denote the natural quotient. Then $d_i = \pi_\Delta \circ d_i^{(1)} = \pi_\Delta \circ (d_i^{(2)})^{-1}$. We denote by $\Sigma_{\Delta 1}, \dots, \Sigma_{\Delta l}$ the connected components of Σ_Δ , we set $\mathcal{P}_\Delta = \pi_\Delta^{-1}(\mathcal{P})$, and we set $\mathcal{P}_{\Delta k} = \mathcal{P}_\Delta \cap \Sigma_{\Delta k}$ for all $1 \leq k \leq l$. We set $\mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta) = \mathcal{M}(\Sigma_{\Delta 1}, \mathcal{P}_{\Delta 1}) \times \dots \times \mathcal{M}(\Sigma_{\Delta l}, \mathcal{P}_{\Delta l})$, and we denote by $\theta_\Delta : \mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta) \rightarrow \mathcal{M}(\Sigma, \mathcal{P})$ the homomorphism induced by π_Δ .

Recall from Subsection 2.7 that, if $h \in \text{Stab}(\Delta)$ and h is pure, then it belongs to $\text{Im}\theta_\Delta$. Moreover, if $h = \theta_\Delta(h_1, \dots, h_l)$, then each h_k is a pure element of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$. This observation is important in the present proof.

Let \tilde{K} be the subgroup of $\mathcal{M}(\Sigma_\Delta, \mathcal{P}_\Delta)$ generated by $\{\tau_{c_1}, \dots, \tau_{c_q}, \tau_{d_0^{(1)}}, \tau_{d_1^{(1)}}, \dots, \tau_{d_p^{(1)}}, \tau_{d_0^{(2)}}, \tau_{d_1^{(2)}}, \dots, \tau_{d_p^{(2)}}\}$, and let $\tilde{K}_k = \tilde{K} \cap \mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$ for $1 \leq k \leq l$. Then \tilde{K}_k is the subgroup of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$ generated by the Dehn twists along the boundary components of $\Sigma_{\Delta k}$, and we have $\tilde{K} = \tilde{K}_1 \times \dots \times \tilde{K}_l$.

Take $(f_1, \dots, f_l) \in \theta_\Delta^{-1}(f)$ and $(g_1, \dots, g_l) \in \theta_\Delta^{-1}(g)$. The equality $\Delta = \Delta(f)$ implies that each f_k is either a pseudo-Anosov element of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$, or a periodic element (namely, an element of \tilde{K}_k). Similarly, each g_k is either a pseudo-Anosov element of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$, or an element of \tilde{K}_k . Furthermore, the equality $f^m = g^m$ implies that $f_k^m \equiv g_k^m \pmod{\tilde{K}_k}$, thus either f_k, g_k are both pseudo-Anosov elements of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$, or they both belong to \tilde{K}_k . Let $h = gf^{-1}$, and, for $1 \leq k \leq l$, let $h_k = g_k f_k^{-1}$. Then $h \in \text{Stab}(\Delta)$ and $\theta_\Delta(h_1, \dots, h_l) = h$. Moreover, since h is pure, each h_k is a pure element of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$. Obviously, if $f_k, g_k \in \tilde{K}_k$, then $h_k = g_k f_k^{-1} \in \tilde{K}_k$.

Assume that f_k, g_k are both pseudo-Anosov elements of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$. Let $\mu_k : \mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k}) \rightarrow \mathcal{M}(\Sigma_{0k}, \mathcal{P}_{\Delta k} \sqcup \mathcal{Q}_k)$ be the corking of $(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$. From the congruence $f_k^m \equiv g_k^m \pmod{\tilde{K}_k}$ we deduce that $\mu_k(f_k)^m = \mu_k(g_k)^m$. By Corollary 2.14, there exist pseudo-Anosov representatives

$F, G \in \text{Diff}(\Sigma_{0k}, \mathcal{P}_{\Delta k} \sqcup \mathcal{Q}_k)$ of $\mu_k(f_k), \mu_k(g_k)$, respectively, such that $F^m = G^m$. Let \mathcal{F}^s (resp. \mathcal{F}^u) be the stable (resp. unstable) foliation of F , and let λ be its dilatation coefficient. Then \mathcal{F}^s (resp. \mathcal{F}^u) is the stable (resp. unstable) foliation of G , and λ is its dilatation coefficient. It follows that $GF^{-1} \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$, thus GF^{-1} has finite order, therefore h_k is periodic (since GF^{-1} is a representative of $\mu_k(h_k)$). Finally, h_k is pure and the only pure periodic elements of $\mathcal{M}(\Sigma_{\Delta k}, \mathcal{P}_{\Delta k})$ are the elements of \tilde{K}_k , thus $h_k \in \tilde{K}_k$.

Let K_0 be the subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ generated by $\{\tau_{c_1}, \dots, \tau_{c_q}, \tau_{d_0}, \tau_{d_1}, \dots, \tau_{d_p}\}$. Note that K_0 is a free abelian group of rank $p + q + 1$ and is contained in the centralizer of f in $\mathcal{M}(\Sigma, \mathcal{P})$. The fact that $h_k \in \tilde{K}_k$ for all $1 \leq k \leq l$ implies that $h \in K_0$, thus

$$g^m = (hf)^m = h^m f^m = h^m g^m,$$

therefore $h^m = 1$. We conclude that $h = 1$ (since K_0 is torsion free), that is, $f = g$. \square

7 Proof of Theorem 3.6

Let Σ be a surface of genus 1 with q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $f, g \in \mathcal{M}(\Sigma)$ such that $f^m = g^m$ for some $m \geq 1$. In particular, $\Delta(f) = \Delta(f^m) = \Delta(g^m) = \Delta(g)$. Set $\Delta = \{\delta_0, \delta_1, \dots, \delta_p\} = \Delta(f) = \Delta(g)$, and choose essential curves $d_0, d_1, \dots, d_p : \mathbb{S}^1 \rightarrow \Sigma$ such that $\langle d_i \rangle = \delta_i$ for all $0 \leq i \leq p$, and $d_i \cap d_j = \emptyset$ for all $0 \leq i \neq j \leq p$. We denote by Σ_{Δ} the surface obtained from Σ cutting along the d_i 's. So, Σ_{Δ} is a compact (but not necessarily connected) surface whose boundary components are $c_1, \dots, c_q, d_0^{(1)}, \dots, d_p^{(1)}, d_0^{(2)}, \dots, d_p^{(2)}$, and $\Sigma = \Sigma_{\Delta} / \sim$, where \sim is the equivalence relation which identifies $d_i^{(1)}(z)$ with $d_i^{(2)}(\bar{z}) = (d_i^{(2)})^{-1}(z)$ for all $0 \leq i \leq p$ and all $z \in \mathbb{S}^1$. We denote by $\Sigma_{\Delta 1}, \dots, \Sigma_{\Delta l}$ the connected components of Σ_{Δ} .

We define a graph $\tilde{\Gamma}$ as follows. The set $\{v_1, \dots, v_l\}$ of vertices is in bijection with the set of connected components of Σ_{Δ} , the set $\{e_0, e_1, \dots, e_p\}$ of edges is in bijection with $\Delta = \{\delta_0, \delta_1, \dots, \delta_p\}$, and an edge e_i connects the vertices v_j and v_k if $d_i^{(1)}$ is a boundary component of $\Sigma_{\Delta j}$ and $d_i^{(2)}$ is a boundary component of $\Sigma_{\Delta k}$. In other words, $\tilde{\Gamma}$ is the dual of the decomposition of Σ determined by d_0, d_1, \dots, d_p . For convenience, we will use the graph Γ which extends $\tilde{\Gamma}$ as follows. We add q vertices v'_1, \dots, v'_q and q edges e'_1, \dots, e'_q to $\tilde{\Gamma}$, and we set that e'_i connects v'_i with v_j if c_i is a boundary component of $\Sigma_{\Delta j}$. So, v'_1, \dots, v'_q are valence 1 vertices of Γ . If $\Sigma_{\Delta j}$ is of genus 1, then v_j may have valence 1, 2, or more. Otherwise, $\Sigma_{\Delta j}$ is of genus 0 and v_j is of valence ≥ 3 in Γ .

We consider four different possible configurations:

Case 1 : one of the components of Σ_{Δ} , say $\Sigma_{\Delta 1}$, is of genus 1 (see Figure 7.1). In this case Γ is a tree.

If all the components of Σ_{Δ} are of genus 0, then $\pi_1(\Gamma)$ is isomorphic to \mathbb{Z} and Γ has a unique reduced cycle. The remaining cases depend on the length of this reduced cycle (length 1, or 2, or more).

Case 2 : there exist $j \in \{0, 1, \dots, p\}$ (say $j = 0$) and a component of Σ_{Δ} (say $\Sigma_{\Delta 1}$) such that $d_0^{(1)}$ and $d_0^{(2)}$ are both boundary components of $\Sigma_{\Delta 1}$ (see Figure 7.2). In this case Γ has a unique reduced cycle which is of length 1. The vertex of this cycle is v_1 and the edge is e_0 .

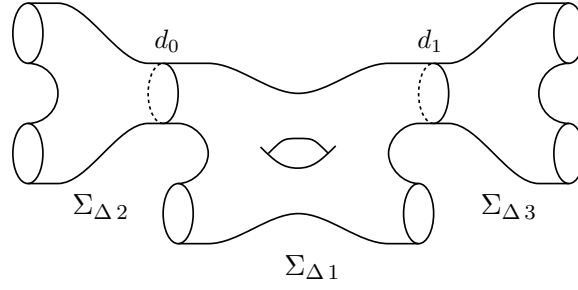


Figure 7.1. The first case.

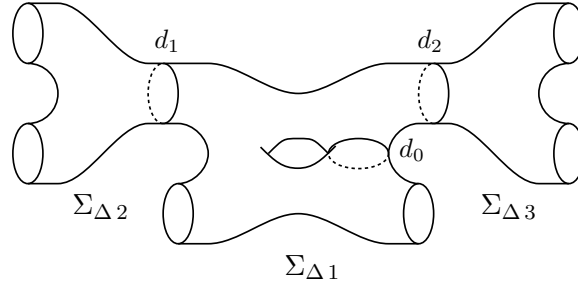


Figure 7.2. The second case.

Case 3 : there exist $j_1, j_2 \in \{0, 1, \dots, p\}$ (say $j_1 = 0$ and $j_2 = 1$) and two components of Σ_{Δ} (say $\Sigma_{\Delta 1}$ and $\Sigma_{\Delta 2}$) such that $d_0^{(1)}$ and $d_1^{(1)}$ are boundary components of $\Sigma_{\Delta 1}$, and $d_0^{(2)}$ and $d_1^{(2)}$ are boundary components of $\Sigma_{\Delta 2}$ (see Figure 7.3). In this case Γ has a unique reduced cycle which is of length 2. The vertices of this cycle are v_1, v_2 and the edges are e_0, e_1 .

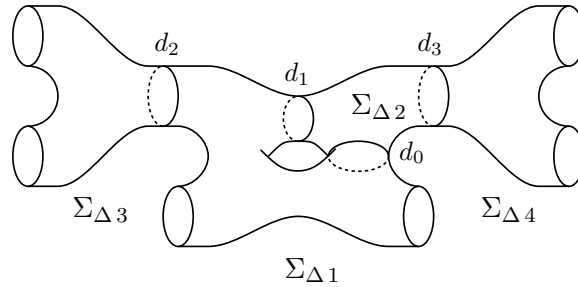


Figure 7.3. The third case.

Case 4 : none of the above (see Figure 7.4). In this case Γ has a unique reduced cycle which is of length ≥ 3 .

The following lemmas are preliminaries. Lemma 7.2 is especially needed to treat Case 1, Lemma 7.3 is needed to treat Case 2, and Lemma 7.4 is needed to treat Case 3. Lemma 7.1 will be used in all the cases.

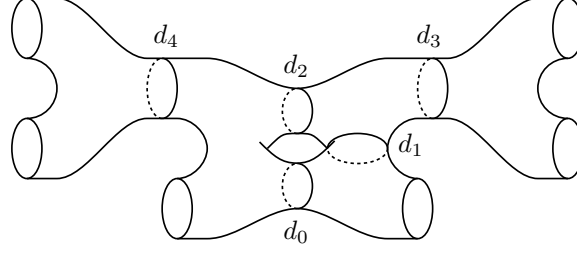


Figure 7.4. The fourth case.

Lemma 7.1. *Let Σ be a surface of genus 0 with q boundary components, c_1, \dots, c_q , where $q \geq 3$. Let K be the subgroup of $\mathcal{M}(\Sigma)$ generated by $\{\tau_{c_1}, \dots, \tau_{c_q}\}$. Let $f, g \in \mathcal{M}(\Sigma)$ be two pseudo-Anosov elements. If $f^m \equiv g^m \pmod{K}$ for some $m \geq 1$, then $f \equiv g \pmod{K}$.*

Lemma 7.2. *Let Σ be a surface of genus 1 with q boundary components, c_1, \dots, c_q , where $q \geq 1$. Let $f, g \in \mathcal{M}(\Sigma)$ be two periodic elements. If $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate.*

Lemma 7.3. *Let Σ be a surface of genus 1 with q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $d : \mathbb{S}^1 \rightarrow \Sigma$ be a non-separating closed curve (see Figure 7.5). Set $\delta = \langle d \rangle$. Let $f, g \in \mathcal{M}(\Sigma)$ such that $\Delta(f) = \Delta(g) = \{\delta\}$. If $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate.*

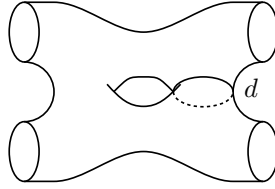


Figure 7.5. A non-separating curve in Σ .

Lemma 7.4. *Let Σ be a surface of genus 1 with q boundary components, c_1, \dots, c_q , where $q \geq 2$. Let $d_1, d_2 : \mathbb{S}^1 \rightarrow \Sigma$ be two non-separating curves such that d_1 is not isotopic to $d_2^{\pm 1}$, and $d_1 \cap d_2 = \emptyset$ (see Figure 7.6). Set $\delta_i = \langle d_i \rangle$ for $i \in \{1, 2\}$, and $\Delta = \{\delta_1, \delta_2\}$. Let $f, g \in \mathcal{M}(\Sigma)$ such that $\Delta(f) = \Delta(g) = \Delta$. If $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate.*

Now, we assume Lemmas 7.1–7.4 (they will be proved later), and turn to finish the proof of Theorem 3.6.

Proof of Theorem 3.6, Case 1. We assume that $\Sigma_{\Delta 1}$ is of genus 1 (see Figure 7.1). For $2 \leq i \leq l$, the surface $\Sigma_{\Delta i}$ is of genus 0 and has at least 3 boundary components. Each $\Sigma_{\Delta i}$ embeds in Σ ($1 \leq i \leq l$), and this embedding determines a homomorphism $\theta_{\Delta i} : \mathcal{M}(\Sigma_{\Delta i}) \rightarrow \mathcal{M}(\Sigma)$ which is injective (see [29]). Let $\pi : \Sigma_{\Delta} \rightarrow \Sigma$ denote the quotient. We set $\mathcal{M}(\Sigma_{\Delta}) =$

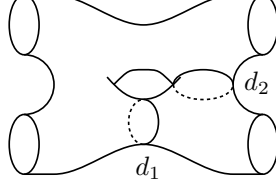


Figure 7.6. Two non-separating curves in Σ .

$\mathcal{M}(\Sigma_{\Delta_1}) \times \mathcal{M}(\Sigma_{\Delta_2}) \times \cdots \times \mathcal{M}(\Sigma_{\Delta_l})$, and we denote by $\theta_{\Delta} : \mathcal{M}(\Sigma_{\Delta}) \rightarrow \mathcal{M}(\Sigma)$ the homomorphism induced by π .

We denote by \tilde{K} the subgroup of $\mathcal{M}(\Sigma_{\Delta})$ generated by $\{\tau_{c_i}; 1 \leq i \leq q\} \cup \{\tau_{d_i^{(1)}}, \tau_{d_i^{(2)}}; 0 \leq i \leq p\}$. This is a free abelian group of rank $q + 2p + 2$. Set $\tilde{K}_i = \tilde{K} \cap \mathcal{M}(\Sigma_{\Delta_i})$ for $1 \leq i \leq l$. Then, for all $1 \leq i \leq l$, \tilde{K}_i is generated by the Dehn twists along the boundary components of Σ_{Δ_i} , and $\tilde{K} = \tilde{K}_1 \times \tilde{K}_2 \times \cdots \times \tilde{K}_l$.

Since $\Delta(f) = \Delta$, the mapping class f induces an isomorphism $f_* : \Gamma \rightarrow \Gamma$. Recall that Γ is a tree and its leafs (i.e. the valence 1 vertices) are v'_1, \dots, v'_q , and possibly v_1 . The curves c_1, \dots, c_q are boundary components of Σ , thus there are fixed by f , therefore $f_*(v'_i) = v'_i$ for all $1 \leq i \leq q$. It is easily proved that that this implies that f_* is the identity on Γ .

The above shows that $f(\delta_k) = \delta_k$ and f preserves the orientation of d_k for all $0 \leq k \leq p$, and Σ_{Δ_j} is invariant under f up to isotopy for all $1 \leq j \leq l$. It follows that there exists $(f_1, f_2, \dots, f_l) \in \mathcal{M}(\Sigma_{\Delta})$ such that $f = \theta_{\Delta}(f_1, f_2, \dots, f_l)$. Furthermore, we have $\Delta(f_i) = \emptyset$ for all $1 \leq i \leq l$, since $\Delta(f) = \Delta = \{\delta_0, \delta_1, \dots, \delta_p\}$. Similarly, there exists $(g_1, g_2, \dots, g_l) \in \mathcal{M}(\Sigma_{\Delta})$ such that $g = \theta_{\Delta}(g_1, g_2, \dots, g_l)$ and $\Delta(g_i) = \emptyset$ for all $1 \leq i \leq l$.

The equality $f^m = g^m$ implies that $f_i^m \equiv g_i^m \pmod{\tilde{K}_i}$ for all $1 \leq i \leq l$. Let $i \in \{2, \dots, l\}$. Since $\Delta(f_i) = \Delta(g_i) = \emptyset$, the elements f_i, g_i are either periodic or pseudo-Anosov. Obviously, the only periodic elements of $\mathcal{M}(\Sigma_{\Delta_i})$ are the elements of \tilde{K}_i . On the other hand, if both, f_i and g_i , are pseudo-Anosov, then, by Lemma 7.1, $f_i \equiv g_i \pmod{\tilde{K}_i}$. So, there exists $\tilde{k} \in \tilde{K}$ such that

$$\theta_{\Delta}(1, g_2, \dots, g_l) = \theta_{\Delta}(1, f_2, \dots, f_l) \cdot \theta_{\Delta}(\tilde{k}).$$

Then

$$g^m \cdot \theta_{\Delta_1}(g_1)^{-m} = \theta_{\Delta}(1, g_2, \dots, g_l)^m = \theta_{\Delta}(1, f_2, \dots, f_l)^m \cdot \theta_{\Delta}(\tilde{k})^m = f^m \cdot \theta_{\Delta_1}(f_1)^{-m} \cdot \theta_{\Delta}(\tilde{k})^m,$$

thus

$$\theta_{\Delta_1}(g_1)^m = \theta_{\Delta_1}(f_1)^m \cdot \theta_{\Delta}(\tilde{k})^{-m}.$$

We know that $g_1^m \equiv f_1^m \pmod{\tilde{K}_1}$, thus $\theta_{\Delta}(\tilde{k})^{-m} \in \theta_{\Delta}(\tilde{K}_1)$. Moreover, $\theta_{\Delta}(\tilde{K})$ is a free abelian group and $\theta_{\Delta}(\tilde{K}_1)$ is a direct factor of $\theta_{\Delta}(\tilde{K})$, hence $\theta_{\Delta}(\tilde{k}) \in \theta_{\Delta}(\tilde{K}_1)$. So, we can assume $\tilde{k} \in \tilde{K}_1$. Set $f'_1 = f_1 \tilde{k}$ and $f'_i = f_i$ for $2 \leq i \leq l$. Then $f = \theta_{\Delta}(f'_1, f'_2, \dots, f'_l)$, $\Delta(f'_1) = \emptyset$, and $\theta_{\Delta_1}(f'_1)^m = \theta_{\Delta_1}(g_1)^m$. Since $\theta_{\Delta_1} : \mathcal{M}(\Sigma_{\Delta_1}) \rightarrow \mathcal{M}(\Sigma)$ is injective, the later equality implies that $f_1'^m = g_1^m$. If f'_1 and g_1 are both pseudo-Anosov, then $f'_1 = g_1$ by Theorem 4.5. If f'_1 and g_1 are both periodic, then they are conjugate by Lemma 7.2. So, f and g are conjugate.

Proof of Theorem 3.6, Case 2. We assume that $d_0^{(1)}$ and $d_0^{(2)}$ are boundary components of $\Sigma_{\Delta 1}$ (see Figure 7.2). Set $\bar{\Sigma}_{\Delta 1} = \Sigma_{\Delta 1} / \sim$, where \sim is the equivalence relation which identifies $d_0^{(1)}(z)$ with $d_0^{(2)}(\bar{z}) = (d_0^{(2)})^{-1}(z)$ for all $z \in \mathbb{S}^1$. Note that $\bar{\Sigma}_{\Delta 1}$ is a surface of genus 1 which embeds in Σ , and this embedding determines a homomorphism $\bar{\theta}_{\Delta 1} : \mathcal{M}(\bar{\Sigma}_{\Delta 1}) \rightarrow \mathcal{M}(\Sigma)$ which is injective (see [29]). For $2 \leq i \leq l$, the surface $\Sigma_{\Delta i}$ is of genus 0 and has at least 3 boundary components. It embeds in Σ , and this embedding determines an injective homomorphism $\theta_{\Delta i} : \mathcal{M}(\Sigma_{\Delta i}) \rightarrow \mathcal{M}(\Sigma)$. Set $\bar{\Sigma}_{\Delta} = \bar{\Sigma}_{\Delta 1} \sqcup \Sigma_{\Delta 2} \sqcup \cdots \sqcup \Sigma_{\Delta l}$, and denote by $\pi : \bar{\Sigma}_{\Delta} \rightarrow \Sigma$ the natural quotient. We set $\mathcal{M}(\bar{\Sigma}_{\Delta}) = \mathcal{M}(\bar{\Sigma}_{\Delta 1}) \times \mathcal{M}(\Sigma_{\Delta 2}) \times \cdots \times \mathcal{M}(\Sigma_{\Delta l})$, and we denote by $\bar{\theta}_{\Delta} : \mathcal{M}(\bar{\Sigma}_{\Delta}) \rightarrow \mathcal{M}(\Sigma)$ the homomorphism induced by π .

Here again, f induces an isomorphism $f_* : \Gamma \rightarrow \Gamma$. Recall that Γ has a unique reduced cycle which is of length 1. The vertex of this cycle is v_1 and the edge is e_0 . In particular, we have $f_*(v_1) = v_1$ and $f_*(e_0) = e_0^{\pm 1}$. Let $\Gamma' = \Gamma \setminus \{e_0\}$. Then f induces an isomorphism $f_* : \Gamma' \rightarrow \Gamma'$, the graph Γ' is a tree, the leafs of Γ' are v'_1, \dots, v'_q , and possibly v_1 , and $f_*(v'_i) = v'_i$ for all $1 \leq i \leq q$, thus f_* is the identity on Γ' .

The above shows that $f(\delta_k) = \delta_k$ for all $0 \leq k \leq p$, f preserves the orientation of d_k for all $1 \leq k \leq p$ (not necessarily for $k = 0$), and $\Sigma_{\Delta j}$ is invariant under f up to isotopy for all $1 \leq j \leq l$. It follows that there exists $(f_1, f_2, \dots, f_l) \in \mathcal{M}(\bar{\Sigma}_{\Delta})$ such that $f = \bar{\theta}_{\Delta}(f_1, f_2, \dots, f_l)$. Furthermore, we have $\Delta(f_1) = \{\delta_0\}$, and $\Delta(f_i) = \emptyset$ for all $2 \leq i \leq l$, since $\Delta(f) = \Delta = \{\delta_0, \delta_1, \dots, \delta_l\}$. Similarly, there exists $(g_1, g_2, \dots, g_l) \in \mathcal{M}(\bar{\Sigma}_{\Delta})$ such that $\bar{\theta}_{\Delta}(g_1, g_2, \dots, g_l) = g$, $\Delta(g_1) = \{\delta_0\}$, and $\Delta(g_i) = \emptyset$ for all $2 \leq i \leq l$. Using the same arguments as in Case 1, it is easily shown that (f_1, f_2, \dots, f_l) can be chosen so that $f_i = g_i$ for all $2 \leq i \leq l$, and $f_1^m = g_1^m$. By Lemma 7.3, it follows that f_1 and g_1 are conjugate in $\mathcal{M}(\bar{\Sigma}_{\Delta 1})$, thus f and g are conjugate.

Proof of Theorem 3.6, Case 3. We assume that $d_0^{(1)}$ and $d_1^{(1)}$ are boundary components of $\Sigma_{\Delta 1}$, and $d_0^{(2)}$ and $d_1^{(2)}$ are boundary components of $\Sigma_{\Delta 2}$ (see Figure 7.3). Set $\bar{\Sigma}_{\Delta 1} = (\Sigma_{\Delta 1} \sqcup \Sigma_{\Delta 2}) / \sim$, where \sim is the equivalence relation which identifies $d_i^{(1)}(z)$ with $d_i^{(2)}(\bar{z}) = (d_i^{(2)})^{-1}(z)$ for all $i \in \{0, 1\}$ and all $z \in \mathbb{S}^1$. Note that $\bar{\Sigma}_{\Delta 1}$ is a surface of genus 1 which embeds in Σ , and this embedding determines a homomorphism $\bar{\theta}_{\Delta 1} : \mathcal{M}(\bar{\Sigma}_{\Delta 1}) \rightarrow \mathcal{M}(\Sigma)$ which is injective. For $3 \leq i \leq l$, the surface $\Sigma_{\Delta i}$ is of genus 0 and has at least 3 boundary components. It embeds in Σ , and this embedding determines an injective homomorphism $\theta_{\Delta i} : \mathcal{M}(\Sigma_{\Delta i}) \rightarrow \mathcal{M}(\Sigma)$. Set $\bar{\Sigma}_{\Delta} = \bar{\Sigma}_{\Delta 1} \sqcup \Sigma_{\Delta 3} \sqcup \cdots \sqcup \Sigma_{\Delta l}$, and denote by $\pi : \bar{\Sigma}_{\Delta} \rightarrow \Sigma$ the natural quotient. We set $\mathcal{M}(\bar{\Sigma}_{\Delta}) = \mathcal{M}(\bar{\Sigma}_{\Delta 1}) \times \mathcal{M}(\Sigma_{\Delta 3}) \times \cdots \times \mathcal{M}(\Sigma_{\Delta l})$, and we denote by $\bar{\theta}_{\Delta} : \mathcal{M}(\bar{\Sigma}_{\Delta}) \rightarrow \mathcal{M}(\Sigma)$ the homomorphism induced by π .

Again, f induces an isomorphism $f_* : \Gamma \rightarrow \Gamma$. Recall that Γ has a unique reduced cycle which is of length 2. The vertices of this cycle are v_1, v_2 and the edges are e_0, e_1 . The graph $\Gamma' = \Gamma \setminus \{e_0, e_1\}$ is the disjoint union of two trees, Γ'_1, Γ'_2 , where v_1 (resp. v_2) is assumed to be a vertex of Γ'_1 (resp. Γ'_2). The vertex v_1 has valence ≥ 3 , thus Γ'_1 has more than one vertex and, equivalently, more than one leaf. We have $f_*(\Gamma') = \Gamma'$ so either $f_*(\Gamma'_1) = \Gamma'_1$, or $f_*(\Gamma'_1) = \Gamma'_2$. We actually have $f_*(\Gamma'_1) = \Gamma'_1$ because Γ'_1 has at least one leaf of the form v'_j and $f_*(v'_j) = v'_j$. All the leafs of Γ'_1 but possibly one, v_1 , are fixed by f_* , thus f_* is the identity of Γ'_1 . Similarly, f_* is the identity on Γ'_2 . In particular, we have $f_*(v_1) = v_1$ and $f_*(v_2) = v_2$.

The above shows that $f(\{\delta_0, \delta_1\}) = \{\delta_0, \delta_1\}$, $f(\delta_k) = \delta_k$ and f preserves the orientation of d_k for all $2 \leq k \leq p$, and $\Sigma_{\Delta j}$ is invariant under f up to isotopy for all $1 \leq j \leq l$. It follows

that there exists $(f_1, f_3, \dots, f_l) \in \mathcal{M}(\bar{\Sigma}_\Delta)$ such that $\bar{\theta}_\Delta(f_1, f_3, \dots, f_l) = f$. Furthermore, we have $\Delta(f_1) = \{\delta_0, \delta_1\}$, and $\Delta(f_i) = \emptyset$ for all $3 \leq i \leq l$, because $\Delta(f) = \Delta = \{\delta_0, \delta_1, \dots, \delta_p\}$. Similarly, there exists $(g_1, g_3, \dots, g_l) \in \mathcal{M}(\bar{\Sigma}_\Delta)$ such that $\bar{\theta}_\Delta(g_1, g_3, \dots, g_l) = g$, $\Delta(g_1) = \{\delta_0, \delta_1\}$, and $\Delta(g_i) = \emptyset$ for all $3 \leq i \leq l$. Using again the same arguments as in Case 1, it is easily shown that (f_1, f_3, \dots, f_l) can be chosen so that $f_i = g_i$ for all $3 \leq i \leq l$, and $f_1^m = g_1^m$. By Lemma 7.4, it follows that f_1 and g_1 are conjugate in $\mathcal{M}(\bar{\Sigma}_{\Delta 1})$, thus f and g are conjugate.

Proof of Theorem 3.6, Case 4. (See Figure 7.4.) For $1 \leq i \leq l$, the surface $\Sigma_{\Delta i}$ is of genus 0 and has at least 3 boundary components. It embeds in Σ , and this embedding determines an injective homomorphism $\theta_{\Delta i} : \mathcal{M}(\Sigma_{\Delta i}) \rightarrow \mathcal{M}(\Sigma)$ (see [29]). Let $\pi : \Sigma_\Delta \rightarrow \Sigma$ denote the natural quotient. We set $\mathcal{M}(\Sigma_\Delta) = \mathcal{M}(\Sigma_{\Delta 1}) \times \mathcal{M}(\Sigma_{\Delta 2}) \times \dots \times \mathcal{M}(\Sigma_{\Delta l})$, and we denote by $\theta_\Delta : \mathcal{M}(\Sigma_\Delta) \rightarrow \mathcal{M}(\Sigma)$ the homomorphism induced by π .

Like in the previous cases, f induces an isomorphism $f_* : \Gamma \rightarrow \Gamma$. Recall that Γ has a unique reduced cycle which is of length ≥ 3 . We assume that the vertices of this cycle are v_1, \dots, v_r and the edges are e_0, e_1, \dots, e_{r-1} . The graph $\Gamma' = \Gamma \setminus \{e_0, e_1, \dots, e_{r-1}\}$ is the disjoint union of r trees, $\Gamma'_1, \dots, \Gamma'_r$, where v_i is assumed to be a vertex of Γ'_i . Since v_i has valence ≥ 3 , the tree Γ'_i has more than one vertex and, equivalently, more than one leaf. We have $f_*(\Gamma') = \Gamma'$ and Γ'_i has a leaf of the form v'_j , thus $f_*(\Gamma'_i) = \Gamma'_i$. Furthermore, all the leafs but possibly one, v_i , are fixed under f_* , thus f_* is the identity on Γ'_i . This also implies that $f_*(v_i) = v_i$. Finally, we have $f_*(e_i) = e_i$ for all $0 \leq i \leq r-1$ because the reduced cycle of Γ is invariant under f_* , all its vertices are fixed by f_* , and its length is ≥ 3 . We conclude that f_* is the identity on the whole graph Γ .

The above shows that $f(\delta_k) = \delta_k$ and f preserves the orientation of d_k for all $0 \leq k \leq p$, and $\Sigma_{\Delta j}$ is invariant under f up to isotopy for all $1 \leq j \leq l$. It follows that there exists $(f_1, f_2, \dots, f_l) \in \mathcal{M}(\Sigma_\Delta)$ such that $f = \theta_\Delta(f_1, f_2, \dots, f_l)$. Furthermore, we have $\Delta(f_i) = \emptyset$ for all $1 \leq i \leq l$ because $\Delta(f) = \Delta = \{\delta_0, \delta_1, \dots, \delta_p\}$. Similarly, there exists $(g_1, g_2, \dots, g_l) \in \mathcal{M}(\Sigma_\Delta)$ such that $\theta_\Delta(g_1, g_2, \dots, g_l) = g$ and $\Delta(g_i) = \emptyset$ for all $1 \leq i \leq l$. Using the same arguments as in Case 1, it is easily shown that (f_1, f_2, \dots, f_l) can be chosen so that $f_i = g_i$ for all $2 \leq i \leq l$, and $f_1^m = g_1^m$. We conclude that $f_1 = g_1$ by Proposition 3.2, hence $f = g$. \square

7.1 Proof of Lemma 7.1.

Lemma 7.5. *Let $H : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a diffeomorphism of the sphere \mathbb{S}^2 of finite order $m \geq 2$. Then H has at most 2 fixed points.*

Proof. Consider the ramified covering $\pi : \mathbb{S}^2 \rightarrow \mathbb{S}^2/H$ and denote by Q_1, \dots, Q_l the ramification points of π . By Lemma 2.27, we have

$$\chi(\mathbb{S}^2) + \sum_{i=1}^l (m - o(Q_i)) = m \cdot \chi(\mathbb{S}^2/H).$$

Suppose that H has at least 3 fixed points. Then $l \geq 3$ and we can suppose $o(Q_1) = o(Q_2) = o(Q_3) = 1$. From the above equality follows

$$3m - 1 = \chi(\mathbb{S}^2) + 3(m - 1) \leq m \cdot \chi(\mathbb{S}^2/H) \leq 2m,$$

and this inequality holds only if $m \leq 1$: a contradiction. \square

Proof of Lemma 7.1. Let Σ be a surface of genus 0 with q boundary components, c_1, \dots, c_q , where $q \geq 3$. Let K be the subgroup of $\mathcal{M}(\Sigma)$ generated by $\{\tau_{c_1}, \dots, \tau_{c_q}\}$. Let $f, g \in \mathcal{M}(\Sigma)$ be two pseudo-Anosov elements such that $f^m \equiv g^m \pmod{K}$ for some $m \geq 1$.

Consider the corking $\theta : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{Q})$ of Σ as defined in Subsection 2.1. By Proposition 2.1, the kernel of θ is precisely K , thus the congruence $f^m \equiv g^m \pmod{K}$ is equivalent to the equality $\theta(f)^m = \theta(g)^m$. Take a pseudo-Anosov representative F of $\theta(f)$ and denote by \mathcal{F}^s (resp. \mathcal{F}^u) the stable (resp. unstable) foliation of F . By Corollary 2.14, there exists a pseudo-Anosov representative $G \in \text{Diff}(\Sigma_0, \mathcal{Q})$ of $\theta(g)$ such that $F^m = G^m$. Set $H = F^{-1}G$. Then $H \in \text{Sym}(\mathcal{F}^s, \mathcal{F}^u)$, a finite group (see Corollary 2.10). Observe furthermore that $H(Q) = Q$ for all $Q \in \mathcal{Q}$ and $|\mathcal{Q}| \geq 3$, thus, by Lemma 7.5, $H = \text{Id}$ and $F = G$. We conclude that $\theta(f) = \theta(g)$, that is, $f \equiv g \pmod{K}$. \square

7.2 Proof of Lemma 7.2

The following lemmas 7.6–7.10 are preliminaries to the proof of Lemma 7.2.

Let Σ be a surface, and let \mathcal{P} be a finite set of punctures in the interior of Σ . Then $\text{PDiff}(\Sigma, \mathcal{P})$ denotes the group of $F \in \text{Diff}(\Sigma, \mathcal{P})$ which pointwise fixed the elements of \mathcal{P} , and $\mathcal{PM}(\Sigma, \mathcal{P})$ denotes the group of isotopy classes of elements of $\text{PDiff}(\Sigma, \mathcal{P})$. Note that $\mathcal{PM}(\Sigma, \mathcal{P})$ is a normal subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ and we have the exact sequence

$$1 \rightarrow \mathcal{PM}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \text{Sym}(\mathcal{P}) \rightarrow 1.$$

Lemma 7.6. *Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ be a set of 4 punctures in the torus \mathbb{T}^2 . Let $f, g \in \mathcal{PM}(\mathbb{T}^2, \mathcal{P})$ be two elements of order 2. Then f and g are conjugate in $\mathcal{PM}(\mathbb{T}^2, \mathcal{P})$.*

Proof. Take a diffeomorphism $F \in \text{PDiff}(\mathbb{T}^2, \mathcal{P})$ of order 2 which represents f , and consider the ramified covering $\pi : \mathbb{T}^2 \rightarrow \mathbb{T}^2/F$. Let Q_1, \dots, Q_l be the ramification points of π . Since F is of order 2, we have $o(Q_i) = |\pi^{-1}(Q_i)| = 1$ for all $1 \leq i \leq l$. Moreover, we can and do assume that $\pi^{-1}(Q_i) = \{P_i\}$ for $1 \leq i \leq 4$. In particular, $l \geq 4$. By Lemma 2.27, we have

$$\chi(\mathbb{T}^2) + l = 2 \cdot \chi(\mathbb{T}^2/F).$$

thus $\chi(\mathbb{T}^2/F) = 2$ (that is, \mathbb{T}^2/F is a sphere), and $l = 4$.

We choose a cellular decomposition of \mathbb{T}^2/F whose vertices are Q_1, Q_2, Q_3, Q_4 , and having 4 arrows, a_1, a_2, a_3, a_4 , and two faces, A_1, A_2 (see Figure 7.7). The source of a_i is assumed to be Q_i and its target to be Q_{i+1} (the indices are taken mod 4). Moreover, we assume $\partial A_1 = a_1 a_2 a_3 a_4$ and $\partial A_2 = a_4^{-1} a_3^{-1} a_2^{-1} a_1^{-1}$. For $1 \leq i \leq 4$, the set $\pi^{-1}(a_i)$ is the union of two edges, b_i and b'_i , such that $b_i \cap b'_i = \{P_i, P_{i+1}\}$. For $j = 1, 2$, the set $\pi^{-1}(A_j)$ is the union of two faces, B_j and B'_j , such that $B_j \cap B'_j = \{P_1, P_2, P_3, P_4\}$. Furthermore, the set $\{P_1, P_2, P_3, P_4\} \cup \{b_1, b'_1, \dots, b_4, b'_4\} \cup \{B_1, B'_1, B_2, B'_2\}$ determines a cellular decomposition of \mathbb{T}^2 . Up to a permutation of either some $\{b_i, b'_i\}$, or some $\{B_j, B'_j\}$, such a cellular decomposition is unique. So, F is unique up to a conjugation in $\text{PDiff}(\mathbb{T}^2, \mathcal{P})$. \square

$B_j \cap B_j'' = B_j' \cap B_j'' = \{P_1, P_2, P_3\}$. Furthermore, $\{P_1, P_2, P_3\} \cup \{b_1, b_1', b_1'', b_2, b_2', b_2'', b_3, b_3', b_3''\} \cup \{B_1, B_1', B_1'', B_2, B_2', B_2''\}$ determines a cellular decomposition of \mathbb{T}^2 . Up to a permutation of either some $\{b_i, b_i', b_i''\}$, or some $\{B_j, B_j', B_j''\}$, such a decomposition is unique. So, F is unique up to a conjugation in $\text{PDiff}(\mathbb{T}^2, \mathcal{P})$. \square

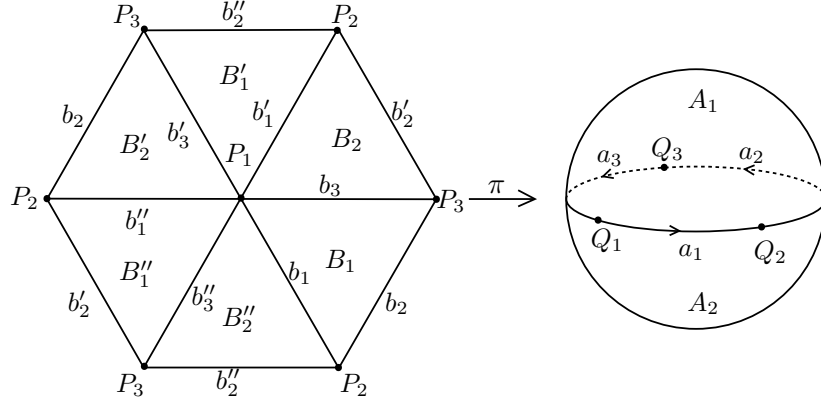


Figure 7.8. Ramified covering associated to an order 3 diffeomorphism of \mathbb{T}^2 .

Lemma 7.9. *Let Σ be a surface of genus 1 with 3 boundary components, c_1, c_2, c_3 . Then there exist $f_0, g_0 \in \mathcal{M}(\Sigma)$ such that $g_0^3 = f_0^2 = \tau_{c_1} \tau_{c_2} \tau_{c_3}$.*

Proof. Consider the curves a_1, a_2, a_3, b pictured in Figure 7.9, and set $f_0 = \tau_{a_1} \tau_b \tau_{a_2} \tau_b \tau_{a_3} \tau_b$ and $g_0 = \tau_{a_1} \tau_{a_2} \tau_{a_3} \tau_b$. Then $g_0^3 = f_0^2 = \tau_{c_1} \tau_{c_2} \tau_{c_3}$. These equalities can be easily checked inspecting the action of g_0^3 (resp. f_0^2) on the isotopy classes of the curves a_1, a_2, a_3, b , and on the isotopy classes of some chosen arcs that join different components of $\partial\Sigma$. This can be found, for instance, in [26]. \square

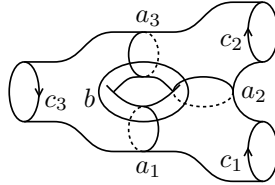


Figure 7.9. A surface of genus 1 with 3 boundary components.

Lemma 7.10. *Let Σ be a surface of genus 1 with 4 boundary components, c_1, c_2, c_3, c_4 . Then there exists $f_0 \in \mathcal{M}(\Sigma)$ such that $f_0^2 = \tau_{c_1} \tau_{c_2} \tau_{c_3} \tau_{c_4}$.*

Proof. Consider the curves a_1, a_2, a_3, a_4, b pictured in Figure 7.10, and set $f_0 = \tau_{a_1} \tau_{a_3} \tau_b \tau_{a_2} \tau_{a_4} \tau_b$. Then $f_0^2 = \tau_{c_1} \tau_{c_2} \tau_{c_3} \tau_{c_4}$. Here again, this equality can be easily checked inspecting the action of f_0^2 on the isotopy classes of the curves a_1, a_2, a_3, a_4, b , and on the isotopy classes of some chosen arcs that join different components of $\partial\Sigma$. \square

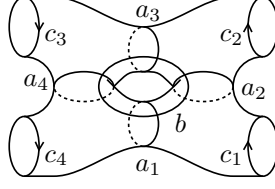


Figure 7.10. A surface of genus 1 with 4 boundary components.

Proof of Lemma 7.2. Let Σ be a surface of genus 1 with q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $f, g \in \mathcal{M}(\Sigma)$ be two periodic elements such that $f^m = g^m$ for some $m \geq 1$.

If $q = 1$, then $\mathcal{M}(\Sigma) \simeq \mathcal{B}_3$, and if $q = 2$, then $\mathcal{M}(\Sigma) = \mathcal{B}_4 \times \mathbb{Z}$ (see [26], [13]). In both cases the conclusion of Lemma 7.2 follows from [14]. So, we can assume $q \geq 3$.

Let $\theta : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{Q})$ be the corking of Σ , as defined in Subsection 2.1. Note that Σ_0 is a torus and $|\mathcal{Q}| = q \geq 3$. Since f and g are periodic, both, $\theta(f)$ and $\theta(g)$, are finite order elements. Moreover, they both belong to $\mathcal{PM}(\Sigma_0, \mathcal{Q})$. We denote by $o(f)$ (resp. $o(g)$) the order of $\theta(f)$ (resp. $\theta(g)$). Let $F \in \text{PDiff}(\Sigma_0, \mathcal{Q})$ be a diffeomorphism of order $o(f)$ which represents $\theta(f)$, and consider the ramified covering $\pi : \Sigma_0 \rightarrow \Sigma_0/F$. The fact that \mathcal{Q} is pointwise fixed under F implies, by Lemma 2.27, that

$$\chi(\Sigma_0) + q(o(f) - 1) \leq o(f) \cdot \chi(\Sigma_0/F).$$

Obviously, this inequality holds only if either $o(f) = 1$, or $o(f) = 2$ and $q \in \{3, 4\}$, or $o(f) = 3$ and $q = 3$. Similarly, we have either $o(g) = 1$, or $o(g) = 2$ and $q \in \{3, 4\}$, or $o(g) = 3$ and $q = 3$.

Set $O(q) = \{1, 2, 3\}$ if $q = 3$, $O(q) = \{1, 2\}$ if $q = 4$, and $O(q) = \{1\}$ if $q \geq 5$. By Lemmas 7.9 and 7.10, for all $r \in O(q)$, there exists $u_r \in \mathcal{M}(\Sigma)$ such that $(u_r)^r = \tau_{c_1} \tau_{c_2} \dots \tau_{c_q}$. By Lemmas 7.6, 7.7 and 7.8, there exist $x_1, \dots, x_q \in \mathbb{Z}$ such that f is conjugate to $u_{o(f)} \cdot \tau_{c_1}^{x_1} \dots \tau_{c_q}^{x_q}$. Thus f^{6m} is conjugate to $\tau_{c_1}^{6mx_1 + \frac{6m}{o(f)}} \dots \tau_{c_q}^{6mx_q + \frac{6m}{o(f)}}$. This implies that

$$f^{6m} = \tau_{c_1}^{6mx_1 + \frac{6m}{o(f)}} \dots \tau_{c_q}^{6mx_q + \frac{6m}{o(f)}},$$

since $\tau_{c_1}^{6mx_1 + \frac{6m}{o(f)}} \dots \tau_{c_q}^{6mx_q + \frac{6m}{o(f)}}$ is central in $\mathcal{M}(\Sigma)$. Similarly, there exist $y_1, \dots, y_q \in \mathbb{Z}$ such that g is conjugate to $u_{o(g)} \cdot \tau_{c_1}^{y_1} \dots \tau_{c_q}^{y_q}$, and

$$g^{6m} = \tau_{c_1}^{6my_1 + \frac{6m}{o(g)}} \dots \tau_{c_q}^{6my_q + \frac{6m}{o(g)}}.$$

Now, the equality $f^{6m} = g^{6m}$ implies that $o(f) = o(g)$ and $x_i = y_i$ for all $1 \leq i \leq q$, thus f and g are conjugate. \square

7.3 Proof of Lemma 7.3

The following lemmas 7.11–7.13 are preliminaries to the proof of Lemma 7.3. Lemma 7.11 can be easily proved using similar arguments as in the proofs of Lemmas 7.5 and 7.8, so its proof is left to the reader. Lemma 7.12 is quite standard, so its proof is also left to the reader.

Lemma 7.11. Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ be a set of 4 punctures in the sphere \mathbb{S}^2 . Let $f, g \in \mathcal{M}(\mathbb{S}^2, \mathcal{P})$ be two elements of order 2 such that $f(P_1) = g(P_1) = P_1$, $f(P_2) = g(P_2) = P_2$, $f(P_3) = g(P_3) = P_4$, and $f(P_4) = g(P_4) = P_3$. Then there exists $h \in \mathcal{PM}(\mathbb{S}^2, \mathcal{P})$ such that $g = hfh^{-1}$. \square

Lemma 7.12. Let Σ be a surface of genus 0 with 2 boundary components, c_1, c_2 , and let $\mathcal{P} = \{P_1, P_2\}$ be a set of 2 punctures in the interior of Σ . Then there exists $f_0 \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $f_0(P_1) = P_2$, $f_0(P_2) = P_1$, and $f_0^2 = \tau_{c_1}\tau_{c_2}$. \square

Lemma 7.13. Let Σ be a surface of genus 0 with q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $\mathcal{P} = \{P_1, P_2\}$ be a set of two punctures in the interior of Σ . Let $f, g \in \mathcal{M}(\Sigma, \mathcal{P})$ be two periodic elements. If $f^m = g^m$ for some $m \geq 1$, then f and g are conjugate.

Proof. If $q = 1$, then $\mathcal{M}(\Sigma, \mathcal{P}) \simeq \mathbb{Z}$. The conclusion of Lemma 7.13 obviously holds in this case, so we can assume $q \geq 2$.

Consider the corking $\theta : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \mathcal{M}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ of (Σ, \mathcal{P}) . Then Σ_0 is a sphere and $|\mathcal{Q}| = q$. Since f and g are periodic, $\theta(f)$ and $\theta(g)$ have finite order. Let $o(f)$ (resp. $o(g)$) denote the order of $\theta(f)$ (resp. $\theta(g)$). Choose a diffeomorphism $F \in \text{Diff}(\Sigma_0, \mathcal{P} \sqcup \mathcal{Q})$ of order $o(f)$ which represents $\theta(f)$, and consider the ramified covering $\pi : \Sigma_0 \rightarrow \Sigma_0/F$. Note that \mathcal{Q} is pointwise fixed under F , thus, by Lemma 2.27,

$$\chi(\Sigma_0) + q(o(f) - 1) \leq o(f) \cdot \chi(\Sigma_0/F).$$

This equality holds only if either $o(f) = 1$, or $q = 2$. If $q = 2$, then $o(f) = 2$ because, in that case, $|\mathcal{P}| = 2$ and \mathcal{P} must be an orbit of F . Similarly, we have either $o(g) = 1$, or $o(g) = 2$ and $q = 2$.

Set $O(q) = \{1, 2\}$ if $q = 2$, and $O(q) = \{1\}$ if $q \geq 3$. By Lemma 7.12, for all $r \in O(q)$, there exists $u_r \in \mathcal{M}(\Sigma, \mathcal{P})$ such that $(u_r)^r = \tau_{c_1} \dots \tau_{c_q}$. By Lemma 7.11, there exist $x_1, \dots, x_q \in \mathbb{Z}$ such that f is conjugate to $u_{o(f)} \cdot \tau_{c_1}^{x_1} \dots \tau_{c_q}^{x_q}$ (recall that, if $o(f) = 2$, then $q = 2$, $\mathcal{P} \sqcup \mathcal{Q}$ has 4 elements, and $\theta(f)$ permutes the elements of \mathcal{P} and fixes the elements of \mathcal{Q}), hence $f^{2m} = \tau_{c_1}^{2mx_1 + \frac{2m}{o(f)}} \dots \tau_{c_q}^{2mx_q + \frac{2m}{o(f)}}$. Similarly, there exist $y_1, \dots, y_q \in \mathbb{Z}$ such that g is conjugate to $u_{o(g)} \cdot \tau_{c_1}^{y_1} \dots \tau_{c_q}^{y_q}$, and $g^{2m} = \tau_{c_1}^{2my_1 + \frac{2m}{o(g)}} \dots \tau_{c_q}^{2my_q + \frac{2m}{o(g)}}$. Finally, the equality $f^{2m} = g^{2m}$ implies that $o(f) = o(g)$ and $x_i = y_i$ for all $1 \leq i \leq q$, thus f and g are conjugate. \square

Proof of Lemma 7.3. Let Σ be a surface of genus 1 with q boundary components, c_1, \dots, c_q , where $q \geq 1$, and let $d : \mathbb{S}^1 \rightarrow \Sigma$ be a non-separating closed curve (see Figure 7.5). We denote by Σ_Δ the surface obtained from Σ cutting along d . So, Σ_Δ is a surface of genus 0 with $q + 2$ boundary components, $c_1, \dots, c_q, d^{(1)}, d^{(2)}$, and $\Sigma = \Sigma_\Delta / \sim$, where \sim is the equivalence relation which identifies $d^{(1)}(z)$ with $d^{(2)}(\bar{z}) = (d^{(2)})^{-1}(z)$ for all $z \in \mathbb{S}^1$. Take two copies \mathbb{D}_1 and \mathbb{D}_2 of the standard disk \mathbb{D} , and denote by $e_i : \mathbb{S}^1 \rightarrow \partial\mathbb{D}_i$ the boundary component of \mathbb{D}_i ($i = 1, 2$). Define the surface $\Sigma_0 = (\Sigma_\Delta \sqcup \mathbb{D}_1 \sqcup \mathbb{D}_2) / \sim$, where \sim is the equivalence relation which identifies $d^{(i)}(z)$ with $e_i(\bar{z}) = e_i^{-1}(z)$ for all $z \in \mathbb{S}^1$ and all $i \in \{1, 2\}$. Note that Σ_0 is a surface of genus 0 with q boundary components, c_1, \dots, c_q . We fix a point Q_i in the interior of \mathbb{D}_i for all $i \in \{1, 2\}$.

We view $\delta = \langle d \rangle$ as a 0-simplex of $\mathcal{C}(\Sigma)$, and we set $\text{Stab}(\delta) = \{h \in \mathcal{M}(\Sigma); h(\delta) = \delta\}$. Let $h \in \text{Stab}(\delta)$. Then we can choose a representative $H \in \text{Diff}(\Sigma)$ of h such that either $H \circ d = d$

(if h preserves the orientation of d), or $H \circ d = d^{-1}$ (if h reverses the orientation of d). Such a diffeomorphism can be left to a diffeomorphism $H_\Delta : \Sigma_\Delta \rightarrow \Sigma_\Delta$ (which does not necessarily belong to $\text{Diff}(\Sigma_\Delta)$), and H_Δ can be extended to a diffeomorphism $H_0 \in \text{Diff}(\Sigma_0, \{Q_1, Q_2\})$, which is unique up to isotopy. Now, the following can be easily proved using classical techniques that can be found, for example, in [19].

Claim 1. *The mapping $h \mapsto H_0$ determines an epimorphism $\mu : \text{Stab}(\delta) \rightarrow \mathcal{M}(\Sigma_0, \{Q_1, Q_2\})$ whose kernel is the cyclic group $\langle \tau_d \rangle$ generated by τ_d .*

Let $f, g \in \mathcal{M}(\Sigma)$ such that $\Delta(f) = \Delta(g) = \{\delta\}$, and such that $f^m = g^m$ for some $m \geq 1$. In particular, $f, g \in \text{Stab}(\delta)$. We have $\Delta(\mu(f)) = \Delta(\mu(g)) = \emptyset$ and $\mu(f)^m = \mu(g)^m$, thus either $\mu(f)$ and $\mu(g)$ are both pseudo-Anosov, or they are both periodic.

Suppose that $\mu(f)$ and $\mu(g)$ are pseudo-Anosov. Then, by Theorem 4.5, $\mu(f) = \mu(g)$. By Claim 1, it follows that there exists $x \in \mathbb{Z}$ such that $g = f\tau_d^x$. Note that τ_d lies in the center of $\text{Stab}(\delta)$, thus $f^m = g^m = f^m\tau_d^{mx}$, therefore $x = 0$ and $f = g$.

Suppose that $\mu(f)$ and $\mu(g)$ are periodic. By Lemma 7.13, there are conjugate. So, there exists $h \in \text{Stab}(\delta)$ such that $\mu(g) = \mu(hfh^{-1})$. Since $\mu(f)$ is periodic, there exist $r \geq 1$ and $y_1, \dots, y_q \in \mathbb{Z}$ such that $\mu(f)^r = \tau_{c_1}^{y_1} \dots \tau_{c_q}^{y_q}$. Furthermore, by Claim 1, there exists $t \in \mathbb{Z}$ such that $f^r = \tau_{c_1}^{y_1} \dots \tau_{c_q}^{y_q} \tau_d^t$. Note that $\tau_{c_1}, \dots, \tau_{c_q}, \tau_d$ lie in the center of $\text{Stab}(\delta)$, thus f^r also lies in the center of $\text{Stab}(\delta)$, therefore $hf^r h^{-1} = f^r$. On the other hand, by Claim 1, there exists $x \in \mathbb{Z}$ such that $g = hf h^{-1} \cdot \tau_d^x$. So,

$$f^{rm} = g^{rm} = (hf h^{-1})^{rm} \tau_d^{rmx} = f^{rm} \tau_d^{rmx},$$

thus $x = 0$ and $g = hf h^{-1}$. □

7.4 Proof of Lemma 7.4

Proof of Lemma 7.4. Let Σ be a surface of genus 1 with q boundary components, c_1, \dots, c_q , where $q \geq 2$. Let $d_1, d_2 : \mathbb{S}^1 \rightarrow \Sigma$ be two non-separating curves such that d_1 is not isotopic to $d_2^{\pm 1}$, and $d_1 \cap d_2 = \emptyset$ (see Figure 7.6). Set $\delta_i = \langle d_i \rangle$ for $i \in \{1, 2\}$, and $\Delta = \{\delta_1, \delta_2\}$. Let Σ_Δ denote the surface obtained from Σ cutting along d_1 and d_2 . So, Σ_Δ is the union of two surfaces of genus 0, $\Sigma_{\Delta 1}$ and $\Sigma_{\Delta 2}$. The boundary components of $\Sigma_{\Delta 1}$ can be assumed to be $c_1, \dots, c_p, d_1^{(1)}, d_2^{(1)}$, and the boundary components of $\Sigma_{\Delta 2}$ to be $c_{p+1}, \dots, c_q, d_1^{(2)}, d_2^{(2)}$, where $1 \leq p \leq q-1$. We have $\Sigma = \Sigma_\Delta / \sim$, where \sim is the equivalence relation which identifies $d_i^{(1)}(z)$ with $d_i^{(2)}(\bar{z}) = (d_i^{(2)})^{-1}(z)$ for all $i \in \{1, 2\}$ and all $z \in \mathbb{S}^1$.

For each $i \in \{1, 2\}$, we take two copies $\mathbb{D}_1^{(i)}$ and $\mathbb{D}_2^{(i)}$ of the standard disk, and we denote by $e_j^{(i)} : \mathbb{S}^1 \rightarrow \partial \mathbb{D}_j^{(i)}$ the boundary component of $\mathbb{D}_j^{(i)}$. We define the surface $\Sigma_{0i} = (\Sigma_{\Delta i} \sqcup \mathbb{D}_1^{(i)} \sqcup \mathbb{D}_2^{(i)}) / \sim$, where \sim is the equivalence relation which identifies $d_j^{(i)}(z)$ with $e_j^{(i)}(\bar{z}) = (e_j^{(i)})^{-1}(z)$ for all $j \in \{1, 2\}$ and all $z \in \mathbb{S}^1$. Note that Σ_{01} (resp. Σ_{02}) is a surface of genus 0 with p (resp. $q-p$) boundary components. We fix a point $Q_j^{(i)}$ in the interior of $\mathbb{D}_j^{(i)}$ for all $i, j \in \{1, 2\}$.

Set $\text{Stab}(\Delta) = \{h \in \mathcal{M}(\Sigma); h(\Delta) = \Delta\}$. Let $h \in \text{Stab}(\Delta)$. We can choose a representative $H \in \text{Diff}(\Sigma)$ of h such that either $H \circ d_1 = d_1$ and $H \circ d_2 = d_2$, or $H \circ d_1 = d_2$ and $H \circ d_2 = d_1$.

Then H determines diffeomorphisms $H_{\Delta i} : \Sigma_{\Delta i} \rightarrow \Sigma_{\Delta i}$, $i \in \{1, 2\}$, and these diffeomorphisms extend to diffeomorphisms $H_{0i} \in \text{Diff}(\Sigma_{0i}, \{Q_1^{(i)}, Q_2^{(i)}\})$, $i \in \{1, 2\}$. Now, in the same manner as Claim 1 in the proof of Lemma 7.3, the following can be easily proved using classical techniques.

Claim 1. Write $\Gamma_i = \mathcal{M}(\Sigma_{0i}, \{Q_1^{(i)}, Q_2^{(i)}\})$ and denote by $\varphi_i : \Gamma_i \rightarrow \text{Sym}_2$ the natural epimorphism. Then the mapping $h \mapsto (H_{01}, H_{02})$ determines a homomorphism $\mu : \text{Stab}(\Delta) \rightarrow \Gamma_1 \times \Gamma_2$. The kernel of μ is the subgroup $\langle \tau_{d_1}, \tau_{d_2} \rangle$ generated by $\{\tau_{d_1}, \tau_{d_2}\}$, which is isomorphic to \mathbb{Z}^2 . The image of μ is $\{(h_1, h_2) \in \Gamma_1 \times \Gamma_2; \varphi_1(h_1) = \varphi_2(h_2)\}$.

Let $f, g \in \mathcal{M}(\Sigma)$ such that $\Delta(f) = \Delta(g) = \Delta$ and $f^m = g^m$ for some $m \geq 1$. Note that $f, g \in \text{Stab}(\Delta)$. Set $(f_1, f_2) = \mu(f)$ and $(g_1, g_2) = \mu(g)$. Then $f_i^m = g_i^m$ and $\Delta(f_i) = \Delta(g_i) = \emptyset$, for $i = 1, 2$. In particular, f_i and g_i are either both pseudo-Anosov, or both periodic.

Claim 2. For $i \in \{1, 2\}$, there exist $\tilde{h}_i \in \mu^{-1}(\Gamma_i)$ and $r_i \geq 1$ such that $g_i = \mu(\tilde{h}_i)f_i\mu(\tilde{h}_i)^{-1}$, $(\varphi_i \circ \mu)(\tilde{h}_i) = 1$, and \tilde{h}_i commutes with $f_i^{r_i}$ for all $f_i \in \mu^{-1}(f_i)$.

We assume $i = 1$. The case $i = 2$ can be treated in the same way. If f_1 and g_1 are both pseudo-Anosov, then, by Theorem 4.5, $f_1 = g_1$. In this case, $\tilde{h}_1 = 1$ and $r_1 = 1$ verify the conclusion of Claim 2. So, we can assume that f_1 and g_1 are periodic. By Lemma 7.11, there exists $h_1 \in \Gamma_1$ such that $\varphi_1(h_1) = 1$ and $g_1 = h_1 f_1 h_1^{-1}$. On the other hand, since f_1 is periodic, there exist $r_1 \geq 1$ and $y_1, \dots, y_p \in \mathbb{Z}$ such that $f_1^{r_1} = \tau_{c_1}^{y_1} \dots \tau_{c_p}^{y_p}$. It follows that, for any $\tilde{f}_1 \in \mu^{-1}(f_1)$, there exist $t_1, t_2 \in \mathbb{Z}$ such that $\tilde{f}_1^{r_1} = \tau_{c_1}^{y_1} \dots \tau_{c_p}^{y_p} \tau_{d_1}^{t_1} \tau_{d_2}^{t_2}$. Clearly, such an element lies in the center of $\{\tilde{h} \in \mu^{-1}(\Gamma_1); (\varphi_1 \circ \mu)(\tilde{h}) = 1\}$. In particular, it commutes with any $\tilde{h}_1 \in \mu^{-1}(h_1)$.

Now, set $h = \tilde{h}_1 \tilde{h}_2$ and $r = r_1 r_2$. Then $h f^r h^{-1} = f^r$ and $\mu(h f h^{-1}) = \mu(g)$. In particular, there exist $x_1, x_2 \in \mathbb{Z}$ such that $h f h^{-1} = g \tau_{d_1}^{x_1} \tau_{d_2}^{x_2}$. Furthermore, since $g \in \text{Stab}(\Delta)$, either $g \tau_{d_1} = \tau_{d_1} g$ and $g \tau_{d_2} = \tau_{d_2} g$ (if $g(\delta_1) = \delta_1$ and $g(\delta_2) = \delta_2$), or $g \tau_{d_1} = \tau_{d_2} g$ and $g \tau_{d_2} = \tau_{d_1} g$ (if $g(\delta_1) = \delta_2$ and $g(\delta_2) = \delta_1$).

Suppose that $g \tau_{d_1} = \tau_{d_1} g$ and $g \tau_{d_2} = \tau_{d_2} g$. Then

$$g^{rm} = f^{rm} = (h f h^{-1})^{rm} = g^{rm} \tau_{d_1}^{rm x_1} \tau_{d_2}^{rm x_2},$$

thus $x_1 = x_2 = 0$ and $g = h f h^{-1}$.

Suppose that $g \tau_{d_1} = \tau_{d_2} g$ and $g \tau_{d_2} = \tau_{d_1} g$. Then

$$(h f h^{-1})^2 = g \tau_{d_1}^{x_1} \tau_{d_2}^{x_2} g \tau_{d_1}^{x_1} \tau_{d_2}^{x_2} = g^2 \tau_{d_1}^{x_1+x_2} \tau_{d_2}^{x_1+x_2},$$

thus

$$g^{2rm} = f^{2rm} = (h f h^{-1})^{2rm} = g^{2rm} \tau_{d_1}^{rm(x_1+x_2)} \tau_{d_2}^{rm(x_1+x_2)},$$

therefore $x_1 + x_2 = 0$. We conclude that

$$\tau_{d_1}^{-x_1} h f h^{-1} \tau_{d_1}^{x_1} = \tau_{d_1}^{-x_1} g \tau_{d_2}^{-x_2} = g \tau_{d_2}^{-x_1-x_2} = g.$$

□

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